MATROIDS ON CHAMBER SYSTEMS

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1. Introduction

Matroids and Coxeter groups. The classical notion of matroid incorporates the concepts of a graph and of linear dependence of vectors, and can be defined by the following surprisingly simple axiom over some system $\mathcal{B}$ of subsets of a finite set E [Aig], [Wel].

> *The pair $M = (E, \mathcal{B})$ is a matroid, if for all A, $B \in \mathcal{B}$ and $x \in A \setminus B$ there exists $y \in B \setminus A$, such that $(A \setminus \{x\}) \cup \{y\}$ lies in $\mathcal{B}$.*

People working in combinatorics know how useful is the notion of matroid. This suggests that, in mathematics, even such a simple object as a finite set should be endowed with some extra structure. The most natural structure on a finite set is provided by its symmetric group acting on it. But, as everybody knows, the symmetric group Sym_n is nothing else but the simplest example of a Coxeter group. WP–matroids in the sense of [GS2] are natural generalizations of matroids for arbitrary finite Coxeter groups W.

A general definition of WP–matroids. Our definition (Section 9) is formulated in terms of the Bruhat ordering on a Coxeter group and can be stated in just a few lines:

> *Let $P < W$ be a parabolic subgroup of a Coxeter group W. A map $\mu\colon W \to W/P$ is called a WP–matroid, if*
>
> $$w^{-1}\mu(u) \leq w^{-1}\mu(w)$$
>
> *for all $u, w \in W$, where $\leq$ is the Bruhat ordering of the left coset space W/P.*

If $W = Sym_n$ is the symmetric group on n letters and

$$P = \langle (1,2), (2,3), \dots, (k-1,k), (k+1,k+2), \dots, (n-1,n) \rangle$$

is the subgroup generated by the transpositions $(i, i+1)$, $1 \leq i \leq n-1$, $i \neq k$, then our definition of a WP–matroid is equivalent to the classical definition of a matroid of rank k on n letters ([GS1], see also discussion in Section 2).

Matroids on thin chamber systems. We would like to suggest also a further generalization of our definition of WP–matroids for Coxeter groups (Section 9). It is formulated in terms of chamber systems.

> *Let C be a thin chamber system (which we identify with its set of chambers, see Section 3). A map $\mu\colon C \to C$ is called a matroid on C, if for all $a, b \in C$*
>
> $$\mu(b) \preceq^a \mu(a),$$
>
> *where $\preceq^a$ is the Bruhat preodering on C with the center a.*

Matroids on flag complexes of triangulations. Now we want to specialize the above definition (all the details of which can be found in Section 9) to a very important case of triangulations of manifolds.

Let T be a triangulation of a n–dimensional manifold and F the set of maximal flags of simplexes in T. (As usually, we can identify maximal flags in T with the corresponding cells of the barycentric subdivision of T). We say that two flags f, $g \in F$ are *i–adjacent*, if they coincide in all dimensions $d \neq i$. Notice that each flag $f \in F$ is i–adjacent to itself for all i, $0 \leq i \leq n$. A *gallery* is a finite sequence of flags $(f_0, \dots, f_m)$, such that f_{k-1} and f_k are adjacent for all k, $1 \leq k \leq m$, m is called the *length* of the gallery. A gallery $(f_0, \dots, f_m)$, connecting flags $f_0 = f$ and $f_m = g$ is called a *geodesic gallery*, if there is no gallery of strictly smaller length with the same property, and is said to be of *type* $i_1 \cdots i_m$, if f_{k-1} and f_k are i_k–adjacent. If f, g, $h \in F$, we say that $g \preceq^h f$ if there is a geodesic gallery

$$(f_0, \dots, f_m), \quad f_0 = h, \quad f_m = f,$$

and a gallery

$$(f'_0, \dots, f'_m), \quad f'_0 = h, \quad f'_m = g,$$

of the same type $i_1 \cdots i_m$, connecting h with the flags f and g, correspondingly.

Now we can give a definition of matroid which will fit in with this particular situation:

A map $\mu\colon F \to F$ is called a matroid on F, if for all f, $g \in F$

$$\mu(g) \preceq^f \mu(f).$$

Convex maps to multiordered sets. A further generalization of the notion of a matroid relates it to combinatorial convexity. Let E be a set with a family of partial orderings $W = \{\leq^w,\ w \in W\}$. We say that a map

$$\mu\colon W \to E$$

is *convex* if it satisfies the inequality

$$\mu(u) \leq^w \mu(w)$$

for all u, $w \in W$.

Sections 2 and 3 contain a discussion of convex maps and their relations to combinatorial convexity and convexity of sets in Euclidean spaces (in the ordinary geometrical meaning of this word).

WP–matroids and Schubert cells. Finally we apply our new understanding of matroids to the combinatorics of flag varieties.

Recall that originally WP–matroids have been introduced by I.M. Gelfand and V. V. Serganova in the works [GS1] and [GS2] for these particular purposes. If W is the Weyl group of a semisimple Lie group G and G_P is a parabolic subgroup in G corresponding to a parabolic subgroup P in W, then WP–matroids describe the partition of the flag variety G/G_P into *thin Schubert cells* [GS2]. It is well known that there is a canonical one-to-one correspondence between Schubert cells on G/G_P and left cosets of W with respect to P. Unfortunately, the partition of G/G_P into Schubert cells is not invariant under the action of the Weyl group W and in this sense depends on the ordering of a coordinate basis. The partition of the flag variety G/G_P into thin Schubert cells is a W–invariant refinement of partitions into Schubert cells. As shown in [GS2], Theorem 2 of Section 8.3 (see also Theorem 6 of the present paper), every thin Schubert cell in G/G_P can be assigned a WP–matroid. The proof of this result in [GS2] is immersed in a more general context of the theory of convex polytopes and moment mappings for compact Kähler manifolds [At], [GMP], [GuS].

But an approach to WP–matroids via chamber systems, first suggested in [BoG], dramatically simplifies and clarifies relations between thin Schubert cells and WP–matroids.

In the paper [BoG] the definition of a WP–matroid has been modified and extended to the case of arbitrary Coxeter groups. This new definition of WP–matroids enabled us to connect them with objects of a more general nature than flag varieties G/G_P, namely, with Tits buildings (Theorems 4 and 6). This approach clarifies and simplifies the situation. We also extend the description of thin Schubert cells on flag varieties of semisimple Lie groups to a more wide context of groups with Tits systems (Theorems 5 and 7). These groups include, in particular, reductive p–adic groups [BrT] and Kac–Moody groups [MoT].

2. Matroids

Definitions and notation are mostly standard and can be found in [Wel] or [BjZ].

2.1. Basic Definitions of Matroid Theory.

Closure operators. A *closure operator* on a set E is an increasing, monotone, idempotent function

$$\tau\colon 2^E \to 2^E$$

on the set 2^E of all subsets in E. This means that for all $A, B \subseteq E$:

(1) $A \subseteq \tau(A)$;
(2) $A \subseteq B$ implies $\tau(A) \subseteq \tau(B)$;
(3) $\tau(\tau(A)) = \tau(A)$.

Matroids: definition in terms of closure operators. A *matroid* or a *finite pregeometry* $M = (E, \tau)$ is a finite set E with a closure operator τ satisfying the Exchange Principle for Closure Operators:

If x, $y \notin \tau(A)$ and $y \in \tau(A \cup \{x\})$, then $x \in \tau(A \cup \{y\})$.

Bases of a matroid. A set $A \subseteq E$ is called *independent*, if $x \notin \tau(A \setminus \{x\})$ for all $x \in A$. Maximal independent sets in E are called *bases* of M. It is easy to prove that all bases of a matroid contain the same number of elements which is called the *rank* of a matroid. The set of all bases of M is called the *base set* of M. Matroids can be characterized in terms of their base sets only.

Fact 1 ([Wel], *Theorem 1.2.1, p. 8*). Let E be a finite set. A set $\mathcal{B} \subseteq 2^E$ of subsets in E is a base set of a matroid if and only if it satisfies the Exchange Principle for Bases:

For all A, $B \in \mathcal{B}$ and $x \in A \setminus B$ there exists $y \in B \setminus A$, such that $(A \setminus \{x\}) \cup \{y\}$ lies in $\mathcal{B}$.

The rank function of a matroid. If M is a matroid on a finite set E, we can define the *rank* $\rho(A)$ of a set $A \subseteq E$ as the maximal number of independent elements in A. This defines the *rank function*

$$\rho\colon 2^E \to \mathbb{Z}.$$

Fact 2 ([Wel], *Theorem 1.2.2, p. 8*). A integer valued function

$$\rho\colon 2^E \to \mathbb{Z}$$

is the rank function of some matroid on E if and only if it satisfies, for all $A \subseteq E$, $x, y \in E$, the following conditions:

- $\rho(\varnothing) = 0$;
- $\rho(A) \leq \rho(A \cup \{x\}) \leq \rho(A) + 1$;
- if $\rho(A \cup \{x\}) = \rho(A \cup \{y\}) = \rho(A)$, then

$$\rho(A \cup \{x\} \cup \{y\}) = \rho(A).$$

2.2. The Dual Matroid. The concept of matroid duality is of fundamental importance in the applications of matroids to combinatorial theory.

The following theorem is due to the founding father of matroid theory, H. Whitney.

Fact 3 (*Whitney* [Whi], [Wel], *Theorem 2.1.1*). Let $\mathcal{B}$ be the base set of a matroid on a finite set E. Then the set

$$\mathcal{B}^* = \{E \setminus B\, ,\, B \in \mathcal{B}\}$$

is the base set of another matroid on E.

Elements of $\mathcal{B}^*$ are called *cobases* of the matroid $\mathcal{B}$ and the matroid $(E, \mathcal{B}^*)$ is called the *dual matroid* of matroid $(E, \mathcal{B})$.

Fact 4 ([Wel], *Theorem 2.1.2*). The rank functions ρ, ρ^* of a matroid $(E, \mathcal{B})$ and the dual matroid $(E, \mathcal{B}^*)$ respectively are related by:

$$\rho^*(E \setminus B) = |E| - \rho(E) - |B| + \rho(B).$$

The function ρ^* is called the *corank function* of a matroid $(E, \mathcal{B})$.

2.3. Matroids arising from Grassmann varieties.

Vector configurations. Let K be a field and $V = K^k$ a k–dimensional vector space over K. Any finite set E of vectors in V is called a *configuration*, if E spans V.

The following fact is well-known.

Fact 5. Let E be a configuration in a k–dimensional vector space V. Then there is a matroid on E of rank k for which the closure operator $\tau\colon 2^E \to 2^E$, the base set $\mathcal{B} \subseteq \mathcal{P}_k(E)$ and the rank function $\rho\colon 2^E \to \mathbb{N}$ are given by:

- $\tau(A) = E \cap \operatorname{span}(A)$ for $A \subseteq E$;
- $B \subseteq E$ lies in $\mathcal{B}$ if B is a base of V;
- $\rho(A) = \dim(\operatorname{span}(A))$ for $A \subseteq E$.

We say that an arbitrary matroid M has a *geometric representation*, if it is isomorphic to a matroid of a vector configuration.

Grassmann varieties and matroids. Let now $G_{n,k}$ be the *Grassmann variety* of k–dimensional vector subspaces in $V = K^n$ with a standard basis

$$E = \{e_1, \dots, e_n\}.$$

Let V^* be the dual space of V with the dual basis

$$E^* = \{e_1^*, \dots e_n^*\}.$$

If now $U \in G_{n,k}$ is a k–dimensional subspace in $V = K^n$, we can assign to U two vector configurations C and C^* of dimension $n-k$ and k, correspondingly. The first of these two configurations, C is the image of the basis E of V in the factor space V/U, and the other configuration one C^* is the image $E^*\Big|_U$ of E^* in the dual space U^* of U with respect to the restriction map

$$\begin{aligned} V^* &\to U^* \\ v^* &\mapsto v^*|_U. \end{aligned}$$

We denote the corresponding matroids on the set $I = \{1, \dots, n\}$ by M_U and M_U^* and call them *matroid* and *comatroid associated with* U.

Proposition 1 (Gelfand–Serganova [GS2]**).** *M^* is the dual matroid of M. The rank functions ρ and ρ^* of M and M^*, correspondingly, are given by the following two equations* (*where for a subset*

$$J = \{i_1, \dots, i_p\} \subseteq I, \quad I = \{1, \dots, n\}$$

we define K^J as the subspace in V spanned by $e_{i_1}, \ldots, e_{i_p}$).

$$\rho(J) = \dim(K^J/K^J \cap U)$$
$$\rho^*(J) = \dim(U/U \cap K^{I\setminus J})$$

Proposition 2. *Assume that the matroid M of rank k on a finite set of n elements has a geometric representation over a field K. Then there exists a k–dimensional subspace $U \in G_{n,k}(K)$ such that the associated comatroid M^*_U is isomorphic to M.*

Proof. Let $C = \{w_1, \ldots w_n\}$ be a configuration of n vectors in k–dimensional vector space $W = K^k$ which provides a geometric realization for M. Let $\{w^*_1, \ldots w^*_k\}$ be any basis in the dual space W^*. Let now $\{e_1, \ldots, e_n\}$ be the canonical basis of K^n. The span of vectors

$$u_i = \sum_{j=1}^{n} w^*_i(w_j)e_j$$

obviously gives the desired k–dimensional subspace U. □

2.4. The Maximality and Minimality Principles.

The Maximality Principle for matroids. Let now $I_n = \{1, 2, \ldots, n\}$ and $\mathcal{P}_k = \mathcal{P}_k(I_n)$ be the set of all k–element subsets in a finite set E. We introduce a partial ordering $\leq$ on $\mathcal{P}_k$ as follows. Let A, $B \in \mathcal{P}_k$, where

$$A = (i_1, \ldots, i_k), \quad i_1 < i_2 < \cdots < i_k$$

and

$$B = (j_1, \ldots, j_k), \quad j_1 < j_2 < \cdots < j_k,$$

then we set

$$A \leq B \iff i_1 \leq j_1, \ldots, i_k \leq j_k.$$

Let $W = \mathrm{Sym}_n$ be the group of all permutations of the elements of I_n. Then we can associate an ordering of $\mathcal{P}_k$ with each $w \in W$ by putting

$$A \leq^w B \iff w^{-1}A \leq w^{-1}B.$$

Clearly $\leq^1$ is just $\leq$.

Fact 6 (*Gale* [Gal], *Gelfand–Serganova* [GS1]). Let $\mathcal{B} \subseteq \mathcal{P}_k$. The set $\mathcal{B}$ is the base set of some matroid if and only if $\mathcal{B}$ satisfies the Maximality Principle:

For every $w \in \mathrm{Sym}_n$ the set $\mathcal{B}$ contains an element $A \in \mathcal{B}$ maximal in $\mathcal{B}$ with respect to $\leq^w$:

$$B \leq^w A \quad \textit{for all} \quad B \in \mathcal{B}.$$

(We call A the w–maximal element in $\mathcal{B}$).

The Minimality principle for matroids. Notice, that in an analogous way one can define the Minimality Principle for matroids:

A set $\mathcal{B} \subseteq \mathcal{P}_k$ satisfies the Minimality Principle, if for every $w \in W$ there is a unique element $A \in \mathcal{B}$ minimal with respect to the ordering $\leq^w$,

$$A \leq^w B \quad \textit{for all} \quad B \in \mathcal{B}.$$

(We call A the w–minimal element of $\mathcal{B}$).

Fact 7 (*Gelfand–Serganova* [GS1]). A set $\mathcal{B} \subseteq \mathcal{P}_k$ is the base set of some matroid if and only if it satisfies the Minimality Principle.

Actually the work [GS1] contains a definition of matroids in terms of the Minimality Principle, but not the Maximality Principle. But obviously the Minimality and the Maximality Principles are equivalent in the sense of the following obvious observation.

Proposition 3. *Let $\mathcal{B}$ and $\mathcal{B}^*$ be the base sets of a matroid and its dual on a finite set E. Then a base $A \in \mathcal{B}$ is w–maximal in $\mathcal{B}$ for $w \in Sym_E$ if and only if the cobase $E \setminus A \in \mathcal{B}^*$ is w–minimal in $\mathcal{B}^*$.*

Matroids on Grassmannians —- an approach via the Maximality Principle. Let now $U \in G_{n,k}$ be a k–dimensional vector subspace in $V = K^n$ and $E = \{e_1, \dots, e_n\}$ the canonical basis in K^n. Let

$$\phi \colon V \to \overline{V} = V/U$$

be the canonical homomorphism of V onto $\overline{V}$. If now $w \in Sym_n$, then we can construct a w–maximal base in the matroid M_U associated with U in the following way. First we chose $i_1 \in I = \{1, \dots, n\}$ such that $\phi(e_{i_1}) \neq 0$ and $w^{-1}(i_1)$ has the maximal possible value. Then if $i_1, \dots,$ i_l are chosen, we take i_{l+1} in $I \setminus \{i_1, \dots, i_l\}$ such that $\phi(e_{i_1}), \dots, \phi(e_{i_{l+i}})$ are linearly independent and $w^{-1}(i_{l+1})$ has the maximal possible value. Since every linearly independent set of vectors can be completed to a base, it produces the w–maximal base of the vector configuration $\phi[E]$ in $\overline{V}$.

Construction of a w–minimal base in the comatroid M_U^* can be described in a much more elementary way. Let

$$E^* = \{e_1^*, \dots, e_n^*\}$$

be the cobasis of the basis E, then for $v \in V$ $e_i^*(v)$ is the i–th coordinate of V in the basis E. Let us take any k linearly independent vectors u_1, $\dots$, u_k in U and form a $n \times k$ matrix

$$A = (u_1, \dots, u_k)$$

of the coordinate column vectors for u_1, $\dots$, u_k. A permutation $w \in Sym_n$ permutes the rows of this matrix. The first k linearly independent rows of the permuted matrix give the w–minimal basis in the configuration $C^* = E^*\Big|_U$ and in the comatroid M^*.

Matroids as maps. Now if $\mathcal{B}$ is the base set of a matroid M on I_n of rank k, we can define a map

$$\mu\colon\ \mathrm{Sym}_n \to \mathcal{P}_k,$$

assigning to each $w \in$ it Sym_n an element $A \in \mathcal{B}$ maximal in $\mathcal{B}$ with respect to $\leq^w$, then this map satisfies the inequality

$$\mu(u) \leq^w \mu(w) \tag{1}$$

for all u, $w \in \mathrm{Sym}_n$. Since any k–set $B \in \mathcal{P}$ can be made maximal in $\mathcal{P}_k$ after some reordering of symbols 1, 2, $\dots$, n, $\mu[\mathrm{Sym}_n] = \mathcal{B}$. Vice versa, the image of every map μ from Sym_n to $\mathcal{P}_k$, satisfying the above inequality, is the base set of some matroid.

In the next section we will study Inequality 1 on its own. We will show that it has a very nice combinatorial interpretation.

3. Combinatorial convexity

Convexity. A *convex hull operator* on a set E is a closure operator τ satisfying the Anti-Exchange Principle:

if $x, y \notin \tau(A)$ *and* $y \in \tau(A \cup x)$, *then* $x \notin \tau(A \cup \{y\})$.

(see Figure 1 for an illustration)

Fact 8 ([BjZ], *p. 321*). Let $\leq$ be an ordering on a set E. Define for $A \subseteq E$

$$\tau(A) = \{x \in E,\, x \leq y \text{ for some } y \in A\}.$$

Then τ is a convex hull operator on E.

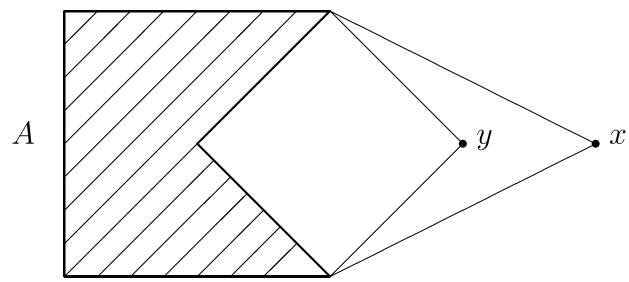

FIGURE 1. Anti-Exchange Principle

Now let W be a family of orderings $\leq^w$, $w \in W$, on a set E, and τ_w the convex hull operator on E constructed from the ordering $\leq^w$. For $A \subseteq E$ set

$$\tau_W(A) = \bigcap_{w \in W} \tau_w(A).$$

Lemma 1. *τ_W is a convex hull operator on E.*

Lemma 1 is an immediate consequence of the following, a slightly more general result.

Lemma 2. *If $\{\tau_i,\, i \in I\}$ is a family of convex hull operators on a set E, then*

$$\tau(A) = \bigcap_{i \in I} \tau_i(A)$$

is also a convex hull operator.

Proof. Firstly we have to check properties (1)–(3) in the definition of a closure operator.

(1) Since $A \subseteq \tau_i(A)$ for all $i \in I$,

$$A \subseteq \bigcap_{i \in I} \tau_i(A) = \tau(A).$$

(2) $A \subseteq B$ implies $\tau_i(A) \subseteq \tau_i(B)$ for all $i \in I$, whence

$$\tau(A) = \bigcap_{i \in I} \tau_i(A) \subseteq \bigcap_{i \in I} \tau_i(B) = \tau(B).$$

(3) Since $\tau(A) \subseteq \tau_i(A)$ for all $i \in I$,

$$\tau_i(\tau(A)) \subseteq \tau_i(\tau_i(A)) = \tau_i(A)$$

and

$$\tau(\tau(A)) = \bigcap_{i \in I} \tau_i(\tau(A)) \subseteq \bigcap_{i \in I} \tau_i(A) = \tau(A).$$

On the other hand, $\tau(A) \subseteq \tau(\tau(A))$ by (1). So $\tau(A) = \tau(\tau(A))$ and τ is a closure operator.

Now we have to check the Anti-Exchange Principle. Assume the contrary, let it fail for $A \subseteq E$ and x, $y \in E$. It means that

$$x, y \notin \tau(A), \quad y \in \tau(A \cup \{x\}), \quad x \in \tau(A \cup \{y\}).$$

Since $x \notin \tau(A)$, $x \notin \tau_k(A)$ for some $k \in I$. Notice

$$y \in \tau(A \cup \{x\}) \subseteq \tau_k(A \cup \{x\})$$

and

$$x \in \tau(A \cup \{y\}) \subseteq \tau_k(A \cup \{y\}).$$

The Anti-Exchange Principle for τ_k yields

$$y \in \tau_k(A).$$

But then

$$A \subseteq A \cup \{y\} \subseteq \tau_k(A),$$

$$\tau_k(A) \subseteq \tau_k(A \cup \{y\}) \subseteq \tau_k(\tau_k(A)) = \tau_k(A)$$

and

$$x \in \tau_k(A \cup \{y\}) = \tau_k(A),$$

a contradiction. $\square$

A classical example. The following lemma gives a natural geometric interpretation of our approach to convexity through orderings.

Let E be a finite dimensional euclidean vector space over the real number field R and E^* its dual space. For any nonzero linear functional $\lambda \in E^*$, $\lambda\colon E \to R$ and vectors x, $y \in E$ we define

$$x \leq^\lambda y \iff \lambda(x) < \lambda(y) \text{ or } x = y.$$

Lemma 3. *Let τ be the convex hull operator associated with the family of orderings*

$$\{\leq^\lambda \mid \lambda \in E^*, \lambda \neq 0\}.$$

on E. Then for any set $A \subseteq E$ we have

$$\tau(A) = A \cup (\mathrm{conv}(A))^\circ,$$

where $\mathrm{conv}(A)$ *is the usual convex hull of A,*

$$\mathrm{conv}(A) = \{\alpha_1 x_1 + \ldots \alpha_k x_k \mid x_1, \ldots, x_k \in A,$$
$$\alpha_1 \geq 0, \ldots, \alpha_k \geq 0,\ \alpha_1 + \cdots + \alpha_k = 1\},$$

and X° denotes the interior of a set $X \subseteq E$.

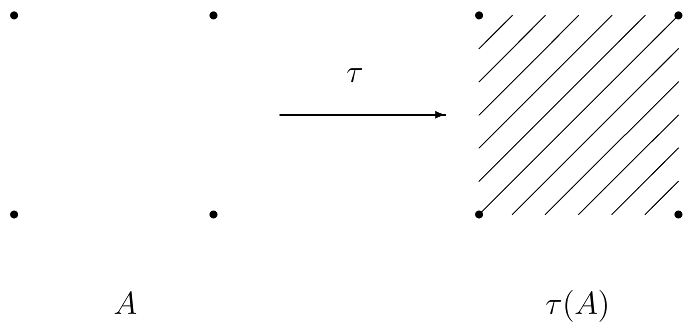

FIGURE 2. Convex hull operator τ on the Euclidean plane

Proof. is obvious (see also the proof of Theorem 3). $\square$

Figure 2 illustrates the action of τ.

Extreme points. For a general closure operator $\tau\colon 2^E \to 2^E$, a point $x \in A$ is called an *extreme point* of $A \subseteq E$ if

$$x \notin \tau(A \setminus \{x\}).$$

We shall denote by $\mathrm{ex}_\tau(A)$ (or by $\mathrm{ex}(A)$ when this brief notation is not misleading) the set of extreme points of A.

Convex maps. Let W be a family of orderings on a set E. A mapping

$$\mu\colon W \to M$$

is called *convex*, if

$$\mu(u) \leq^w \mu(w) \tag{2}$$

for all u, $w \in W$.

Theorem 1. *Let τ_W be the convex hull operator associated with a family W of orderings on a set E, and*

$$\mu\colon W \to E$$

a convex map. Then

$$\mu[W] = \mathrm{ex}(\tau_W(\mu[W])).$$

Proof. Obvious. $\square$

4. Bruhat convexity on chamber systems

Chamber systems. Our exposition of chamber systems follows [Ron]. A set C is a *chamber system* over a set I if each element i of I determines a partition of C, two elements in the same parts being called *i–adjacent.* Thus i–adjacency is an equivalence relation on C. The classes of i–adjacency are called *panels of type* i and the elements of C are called *chambers.* If π is a panel and x is a chamber in π, we shall say, abusing the language, that π is a panel of x. For i–adjacent chambers x and y we shall write $x \sim_i y$. A *gallery* is a finite sequence of chambers $(c_0, \dots, c_k)$ such that c_{j-1} is adjacent to c_j for each $1 \le j \le k$, k is called the *length* of the gallery. The gallery *stammers*, if for some j, $c_{j-1} = c_j$. The gallery is said to be of *type* $i_1 i_2 \cdots i_k$ (a word in the free monoid on I), if c_{j-1} is i_j–adjacent to c_j. A gallery $(c_0, \dots, c_k)$, connecting $x = c_0$ and $y = c_k$ is called a *geodesic gallery*, if there is no gallery of strictly smaller length with the same property. The length of a geodesic gallery from x to y is called the *distance* between x and y and denoted $d(x, y)$.

A *morphism* $\phi\colon C \to D$ between two chamber systems over the same indexing set I is a map preserving the i–adjacency for each $i \in I$ (thus if x and y are i–adjacent, then $\phi(x)$ and $\phi(y)$ are too). The terms *isomorphism* and *automorphism* have the obvious meaning.

Bruhat ordering on thin chamber systems. Let C be a thin chamber system over I (i.e. each panel in A is adjacent to at most two chambers in A) and a, b, c chambers in C. We say that $a \preceq^c b$, if there is a geodesic gallery Γ stretched from c to b and a gallery Γ' of the same type $i_1 \cdots i_k$ connecting c and a. We call $\preceq^c$ the *Bruhat preodering* of C with the center c. Its transitive closure $\le^c$ is called the *strong Bruhat ordering* or, for brevity, the *Bruhat ordering* on C with the center c.

It is easy to see (Lemma 4) that for the Coxeter complex W the strong Bruhat ordering with the center w coincides with a translation of the Bruhat ordering in the classical meaning of these words:

$$u \le^w v \iff w^{-1}u \le w^{-1}v.$$

Weak Bruhat orderings on chamber systems. Let C be a chamber system (not necessarily thin) and c a chamber in C. We say that $a \le^c b$, if a chamber a belongs to some geodesic gallery stretched from

c to b. Obviously this defines a partial ordering on C which is called the *weak Bruhat ordering on* C with the center c.

The Bruhat convexity on thin chamber systems. The convex closure operator τ on a thin chamber complex C associated with a family

$$\{\leq^c \mid c \in C\}$$

of all strong Bruhat orderings on C, is called the *Bruhat convex hull operator on* c, or, for brevity, the *Bruhat convexity* or the *strong convexity* on C.

In the notation of the previous paragraph, we say that the map $\mu\colon C \to C$ is *convex*, if

$$\mu(b) \leq^a \mu(a) \tag{3}$$

for all a, $b \in C$.

Chamber matroids. If we replace in Inequality 3 the ordering $\leq^a$ by the preordering $\preceq^a$, we come to the definition of a *chamber matroid*, as given in [BoG]. If C is the Coxeter complex for a Coxeter group W then the Bruhat preordering coincides with the Bruhat ordering [**Coh**] and, clearly, W–matroids in the sense of [BoG] are convex maps $W \to W$.

Weak convexity. Finally, an analogous definition for the family $\{\leq^c \mid c \in C\}$ of the weak Bruhat orderings on an arbitrary (not necessary thin) chamber system C produces a notion of the *weak Bruhat convex hull operator* or simply *weak convexity* on C.

The following simple characterization of weak convexity ia almost obvious.

Theorem 2. *Let C be a chamber system. A subset $A \subseteq C$ is weakly convex in C if and only if A contains, with every of its chambers a, b, any geodesic gallery strectched from a to b in C.*

Convex hull operators and convex maps on flag complexes of triangulations. Now we want to specialize the above definition to a very important case of triangulations of manifolds.

Let T be a triangulation of a n–dimensional manifold and F the set of maximal flags of simplexes in T. (As usually, we can identify maximal flags in T with the corresponding cells of the barycentric subdivision of T). We say that two flags $f, g \in F$ are *i–adjacent*, if they coincide in all dimensions $d \neq i$. Notice that each flag $f \in F$ is i–adjacent to itself for all i, $0 \leq i \leq n$. A *gallery* is a finite sequence of flags $(f_0, \ldots, f_m)$, such that f_{k-1} and f_k are adjacent for all k, $1 \leq k \leq m$, m is called the *length* of the gallery. A gallery $(f_0, \ldots, f_m)$, connecting flags $f_0 = f$ and $f_m = g$ is called a *geodesic gallery*, if there is no gallery of strictly smaller length with the same property, and is said to be of *type* $i_1 \cdots i_m$, if f_{k-1} and f_k are i_k–adjacent. If f, g, $h \in F$, we say that $g \preceq^h f$, if there is a geodesic gallery

$$(f_0, \ldots, f_m), \quad f_0 = h, \quad f_m = f,$$

and a gallery

$$(f'_0, \ldots, f'_m), \quad f'_0 = h, \quad f'_m = g,$$

of the same type $i_1 \cdots i_m$, connecting h with the flags f and g, correspondingly.

We define an ordering $\leq^h$ as the transitive closure of a preordering $\preceq^h$ and a convexity hull operator τ as the operator associated with a family of orderings $\{\leq^f \mid f \in F\}$.

We can give a definition of a convex map which will fit in with this particular situation:

A map $\mu\colon F \to F$ is convex, if for all f, $g \in F$ $\mu(g) \leq^f \mu(f)$.

Certainly, i–adjacency turns F to a chamber system and our ordering $\leq^f$ and the convex hull operator τ are nothing else but specializations of the Bruhat ordering and the Bruhat convexity of this chamber system.

5. Convex maps from groups to ordered sets

An action of a group G on an ordered set $(E, \leq)$ generates a family $\{\leq^g \mid g \in G\}$ of orderings on E by setting

$$x \leq^g y \iff g^{-1}x \leq g^{-1}y$$

for $x, y \in E$ and gives rise to the convex hull operator τ_G associated with this family of orderings and the notion of a convex map

$$\mu\colon G \to E.$$

Notice, that WP–matroids on a Coxeter group W in the sense of [BoG] are precisely convex maps from W to the factorset W/P modulo a parabolic subgroup $P < W$ and the Bruhat ordering $\leq$ on W/P.

The following observation shows close relations between convex maps on orthogonal groups and strongly convex closed surfaces in Euclidean spaces:

Theorem 3. *Let $G = SO_n$ be the group of rotations of the n–dimensional Euclidean vector space $E = E_n$. Let λ be a nonzero linear functional $\lambda\colon E \to \mathbb{R}$. Consider the following ordering on E: if x, $y \in E$, we set*

$$x \leq y \iff \lambda(x) < \lambda(y) \text{ or } x = y.$$

Then the image $\mu[G]$ of a convex map $\mu\colon G \to E$ either consists of a single point or lies on a convex closed surface

$$S_\mu = \delta(\mathrm{conv}(\mu[G])).$$

If, in addition, the map μ is continuous, then $\mu[G] = S_\mu$ and S_μ is a strongly convex closed surface. Moreover, every closed strongly convex surface in E corresponds in this way to some continuous convex map from G to E.

Here a *convex surface* is the boundary $S = \delta(Q)$ of a convex set Q with a non-empty interior Q° (cf. [Bus]); S is called *closed* if Q is bounded. A *supporting hyperplane* of S (or Q) is a hyperplane (i.e. a linear variety of codimension 1 in E) which contains points of $\delta(Q)$ but not points of Q°. A convex surface is called *strongly convex*, if it has exactly one common point with each of its supporting hyperplane. A *supporting half space* of the convex set Q is a closed half space bounded by a supporting hyperplane of Q and containing Q. It is well known that for a convex set $Q \neq E$, its topological closure $\overline{Q}$ equals the intersection of all supporting closed half spaces of Q [Bus, (1.9), p. 5]. We denote by $\mathrm{conv}(X)$ the convex hull of a set $X \subset E$. If $\overline{\mathrm{conv}(X)}$ is its topological closure, then it is equal to the intersection of all closed subspaces containing X.

The nature of the correspondence between convex mappings and closed strongly convex surfaces is obvious from the following very simple proof of Theorem 3.

Set

$$H = \{x \mid \lambda(x) \leq 0\};$$

this is clearly a closed half space whose boundary $\delta(H)$ is the hyperplane $\lambda(x) = 0$ and the interior H° of H is given by

$$H^\circ = \{x \mid \lambda(x) < 0\}.$$

Our definition of the order $\leq$ can be stated now in the following form:

$$x \leq y \iff x \in (y + H^\circ) \cup \{y\}.$$

It follows from this definition, that if $x \leq a$ for all $x \in X \subset E$, then $X \subseteq a + H^\circ \cup \{a\}$ and, since the latter is convex,

$$\mathrm{conv}(X) \subseteq (a + H^\circ) \cup \{a\}.$$

In particular, if all $x \leq a$ for all $x \in X$, then the same is true for all elements $x' \in conv(X)$: $x' \leq a$.

We say that two closed half spaces in E have *the same orientation*, if one of them contains the another one, or, equivalently, one of them is a parallel translation of another one. The set of all orientations of half spaces in E can be identified in the usual way with the unit sphere

$$U = \{x \in E \mid \|x\| = 1\}$$

and, in particular, is compact. The group $G = O_n$ acts continuously and transitively on the set of all orientations of closed half spaces in E.

We shall also use the following obvious property of convex sets: if two supporting closed half spaces of a convex set have the same orientation, then they coincide.

After these preliminary remarks we can start the proof. The definition of a convex map can be rewritten under the assumptions of the theorem as follows:

$$g^{-1}\mu(h) \leq g^{-1}\mu(g)$$

is equivalent to

$$g^{-1}\mu(h) \in (g^{-1}\mu(g) + H^\circ) \cup \{g^{-1}\mu(g)\},$$

which, in turn, yields

$$\mu(h) \in (\mu(g) + gH^\circ) \cup \{\mu(g)\} \subset \mu(g) + gH.$$

Since these inclusions are valid for all $g, h \in G$, we can take the intersection of their right parts over all $g \in G$ and get

$$\mu[G] \subseteq \bigcap_{g \in G} (\mu(g) + gH).$$

Denote the right part of the last inclusion by Q. Clearly Q is convex as it is an intersection of closed half spaces and $\mathrm{conv}(\mu[G]) \subseteq Q$. Moreover, the set of closed half spaces $\{\mu(g) + gH \mid g \in G\}$ contains closed half spaces of all possible orientations, so Q is obviously equal to the intersection of *all* closed half spaces containing $\mu[G]$, whence equal $\overline{conv(\mu[G])}$. If Q° is empty, then Q lies on some hyperplane P. We can choose $g \in G$ such that $g\delta(H)$ has the same orientation as P, then obviously

$$\mu[G] \subseteq Q \subseteq \mu(g) + g\delta(H),$$

and since

$$\mu[G] \subseteq (\mu(g) + gH^\circ) \cup \{\mu(g)\}$$

by the definition of a convex map, we have $\mu[G] = \{\mu(g)\}$, whence μ is a constant map.

So we can assume that Q° is not empty. Obviously Q is bounded and $S = \delta(Q)$ is a closed convex surface. Each point $\mu(g)$ of $\mu[G]$ lies on the boundary $\mu(g) + g\delta(H)$ of the supporting closed half space $\mu(g) + gH$ of Q, so $\mu[G] \subseteq \delta(Q)$. Thus the first part of the theorem is proved.

Assume now that μ is a continuous mapping. Let $\mu(g) \neq \mu(g')$ be two different points of $\mu[G]$, then the half spaces gH and $g'H$ can not have the same orientation, because in this case the supporting half spaces $\mu(g) + gH$ and $\mu(g') + g'H$ for Q coincide and by definition of the ordering $\leq$ we have $\mu(g) \leq \mu(g')$ and $\mu(g') \leq \mu(g)$, whence $\mu(g) = \mu(g')$. This means that the map $\mu\colon G \to \mu[G] \subseteq S$ can be passed through the unit sphere U (which is identified with the set of all orientations of half spaces in M):

$$\begin{tikzcd} G \arrow[r, "\mu"] \arrow[d] & S \\ U \arrow[ur, "\overline{\mu}"'] & \end{tikzcd}$$

Since S is homeomorphic to U by [Bus, (1.4), page 3] and $\overline{\mu}$ is continuous, it follows from the Borsuk-Ulam Theorem [Mas, Corollary 9.3, p. 170], that either $\overline{\mu}[U] = S$ or two antipode points u, u' of U are mapped to the same point of S. Assume the latter; if now g, g' are elements of G covering u, u', correspondingly, then the supporting half spaces $\mu(g)+gH$ and $\mu(g')+g'H$ for Q have opposite orientations, thus Q lies in their common bounding hyperplane. But then Q° is empty, contrary to a previously made assumption. So we have proved that $\mu[G] = S$. Moreover, now it is clear that S is strongly convex.

If now S is any closed strongly convex surface in E, then clearly the mapping which sends an element $g \in G$ to a (unique) point of intersection with S of the bounding hyperplane of a (unique) supporting half space to S with the same orientation as gH, is a continuous convex map from G to E. $\square$

Notice that easy examples show that one can not skip the continuity assumption from the statement of Theorem 3: there exist (non-continuous) convex mappings $\mu\colon G \to E$ such that $\mu[G] \neq S_\mu$ and S_μ is not strongly convex.

6. W–matroids as convex maps on Coxeter groups

Definitions and notations in this section are mostly standard and may be found in [Bou], [**Coh**] and [Ron].

In this section we introduce W–matroids, a special case of WP–matroids (it corresponds to $P = 1$).

6.1. Coxeter groups.

Definition of Coxeter groups. We recall that a *Coxeter group* is a group W with a finite set of generators R, subject to the relations: $r^2 = 1$ for all $r \in R$, $(r_i r_j)^{m_{ij}} = 1$ for all $r_i, r_j \in R$, where $m_{ij} \in \mathbb{N} \cup \infty$. The set R is called the *set of distinguished generators* for W.

Let $w \in W$. The minimal number of factors in a factorization $w = r_1 \cdot \cdots \cdot r_l$, where $r_i \in R$, is called the *length* of w and is denoted by $l(w)$. A factorization $w = r_1 \cdot \cdots \cdot r_l$ of the least length $l = l(w)$ is called a *reduced expression* for w.

Bruhat ordering. The *Bruhat partial ordering* on W is defined as follows: $u \leq v$, if and only if there is a reduced expression $r_1 \cdot \cdots \cdot r_q$ ($r_j \in R$) for v such that $u = r_{i_1} \cdot \cdots \cdot r_{i_m}$ (where $1 \leq i_1 < i_2 < \cdots < i_m \leq q$). We emphasize that this defines not just a preordering, but an ordering on W (see [**Coh**] or [Bj1] for discussion and proof.) We also define the *weak Bruhat ordering* or the *weak ordering* on W: $v \leq u$ if there exist s_1 and s_2 in W such that $u = s_1 v s_2$ and $l(u) = l(s_1) + l(v) + l(s_2)$.

6.2. W–matroids as convex maps.

W–matroids. A convex map $\mu\colon W \to W$ from a group W to a partially ordered set $(W, \leq)$ has been called in our previous work [BoG] a *W–matroid*. Recall that it means that for all $u, w \in W$ we have

$$w^{-1}\mu(u) \leq w^{-1}\mu(w).$$

Flag W–matroids. For each $w \in W$ we associate a new ordering on W thus: $u \leq^w v$ if $w^{-1}u \leq w^{-1}v$. Clearly the Bruhat ordering introduced above coincides with $\leq^1$.

Let L be an arbitrary subset of W. An element $s \in L$ is called w–maximal in L if $u \leq^w s$ for all $u \in L$. We shall say that L satisfies the *maximality condition*, if for any $w \in W$ there is a w–maximal element in L. In an analogous way one can define the minimality condition [GS2].

Now assume that W is finite. Then it is well known (see [Bou], Exercise IV.1.22), that W contains the longest element w_0. The element w_0 is an involution and the multiplication by w_0 reverses the Bruhat ordering (see, for example, [Bj1], (4.2)),

$$u \leq v \iff w_0 u \geq w_0 v \iff u w_0 \geq v w_0.$$

So for a finite Coxeter group W the minimality condition for a subset $L \subseteq W$ in the sense of [GS2] is equivalent to the maximality condition. *Flag W–matroids* have been introduced in [GS2] as subsets $M \subseteq W$ satisfying the minimality, or, what is equivalent, the maximality condition.

The following proposition is obvious, but fundamental.

Proposition 4. *If W is a finite Coxeter group, then the image $M = \mu[W]$ of a W–matroid $\mu\colon W \to W$ is a flag W–matroid. Conversely, if M is a flag W–matroid and $\mu(w)$ is the w–maximal element in M, then the function $\mu\colon W \to W$ is a W–matroid and $\mu[W] = M$.*

The Bruhat Convexity on Coxeter groups. The action of a Coxeter group W on itself by left shifts

$$w\colon u \mapsto w^{-1} u$$

and the Bruhat ordering on W give rise to the convexity hull operator

$$\tau\colon 2^W \to 2^W$$

(see Section 5). We shall call τ the *Bruhat convexity hull operator* on W.

In the case of finite Coxeter groups the operator τ does not bear any additional information on combinatorics of the Bruhat ordering. Indeed, the following Proposition shows that for finite W the operator τ is trivial.

Proposition 5. *If W is finite, τ is the identity map,*

$$\tau(A) = A$$

for all $A \subseteq W$.

Proof. Let w_0 be the longest element in W, $A \subseteq W$ and $x \in W \setminus A$. Since w_0 is the maximal element in W with respect with Bruhat ordering $\leq$ on W and $w_0 = w_0^{-1}$,

$$w_0 x^{-1} \cdot y \leq w_0 = w_0 x^{-1} \cdot x$$

for all $y \in W$, which means

$$y \leq^{xw_0} x.$$

So x is xw_0–maximal element in W and so $x \notin \tau_{xw_0}(A)$, hence $x \notin \tau(A)$. This proves that $\tau(A) = A$. $\square$

We can also define the *weak Bruhat convexity* on a Coxeter group W by analogy with the Bruhat convexity. The same arguments yield that in the case of finite Coxeter group the corresponding convex hull operator is also trivial.

In the same time easy examples show that the Bruhat convexity on *infinite* Coxeter groups is non-trivial. We hope that the Bruhat convexity may be a usefull tool for study of combinatorics of infinite Coxeter groups. Unfortunately we are at the very beginning of a systematic exploration of convexity properties of Coxeter groups, and we do not have a clear picture even for simplest examples, say, for the affine Coxeter group $\tilde{A}_2$ given by involutive generators r_1, r_2, r_3 and relations

$$(r_1 r_2)^3 = (r_2 r_3)^3 = (r_3 r_1)^3.$$

7. Buildings

We use the approach to buildings developed in [Ti2] and [Ti3]. It is equivalent to the definition given by Tits in [Ti1], see also [Bou]. The exposition of the theory of buildings in [Ron] is the most convenient for our purposes.

7.1. Coxeter complexes.

The Coxeter complex of a Coxeter group. Let W be a Coxeter group with a distinguished set of generators $R = \{r_i, \ i \in I\}$. Take the elements of W as chambers and for each $i \in I$, define i–adjacency by

$$w \sim_i w, \ wr_i.$$

This gives a chamber system over I, which is called the *Coxeter complex* of W. A gallery $\Gamma = (x_0, \ldots, x_k)$ of type $i_1 \cdots i_k$ in W is called *reduced*, if $r_{i_1} \cdot \cdots \cdot r_{i_k}$ is a reduced expression for an element in W.

Geometric interpretation of the Bruhat ordering. The following lemma describes the Bruhat ordering of W in terms of its Coxeter complex. It shows that the Bruhat ordering on W coincides with the strong Bruhat ordering on the Coxeter complex of W and gives us a proof that the Bruhat preordering on a Coxeter complex coincides with the (strong) Bruhat ordering.

Lemma 4. *Let $u, v, w \in W$. We say that $u \succeq^w v$, if there is a geodesic gallery*

$$\Gamma = (x_0, \dots, x_k), \quad w = x_0, \quad x_k = u,$$

and a gallery

$$\Gamma' = (y_0, \dots, y_k), \quad w = y_0, \quad y_k = v,$$

of the same type $i_1 \cdots i_k$, connecting w with the elements u and v, correspondingly. Then

$$u \succeq^w v \iff w^{-1}u \geq w^{-1}v.$$

Proof. $\Longrightarrow$. Clearly the action of W on itself by the left multiplication preserves i–adjacency. If we replace the galleries Γ and Γ' by

$$w^{-1}\Gamma = (w^{-1}x_0, \dots, w^{-1}x_k)$$

and

$$w^{-1}\Gamma' = (w^{-1}y_0, \dots, w^{-1}y_k),$$

correspondingly, we can reduce the proof to the case $w = 1$. By definition of a type of a gallery

$$x_{j-1} \sim_{i_j} x_j,$$

which means

$$x_j = \begin{cases} x_{j-1} & \text{if } \Gamma \text{ stammers at } x_{j-1},\, x_j \\ x_{j-1}r_{i_j} & \text{if not,} \end{cases}$$

for all $1 \leq j \leq k$. Obviously the geodesic gallery Γ does not stammer at any its chamber. So for all $1 \leq j \leq k$ we have

$$x_j = x_{j-1}r_{i_j}$$

and, since $x_0 = 1$, we have $x_k = r_{i_1} \cdot \cdots \cdot r_{i_k}$. Moreover, this is clearly a reduced expression for x_k.

If we repeat now the same arguments for Γ', we have

$$y_j = \begin{cases} y_{j-1} \\ y_{j-1}r_{i_j} \end{cases}$$

and

$$y_k = r_{i_\alpha} r_{i_\beta} \cdot \cdots \cdot r_{i\omega}$$

for some sequence of indexes $1 \leq \alpha < \beta < \cdots < \omega \leq k$. But by definition of the Bruhat ordering it means that

$$y_k \leq x_k.$$

The implication $\Longleftarrow$ is obvious. $\square$

The geometric interpretation of the weak ordering. The following simple fact on the weak ordering of a Coxeter group is well known.

Fact 9 (*Bjorner* [Bj1]*, p.176*). Let $\leq$ denotes the weak ordering on a Coxeter group W and

$$u \leq^w v \iff w^{-1}u \leq w^{-1}v.$$

Then $u \leq^w v$ if and only if u lies in some geodesic gallery stretched from w to v.

7.2. Buildings. Let W be a Coxeter group with the distinguished set of generators $R = \{r_i,\, i \in I\}$. By definition, *building of type* W is a chamber system Δ over I such that each panel belongs to at least two chambers, and having a W*–distance function*

$$\delta \colon \Delta \times \Delta \to W,$$

such that if $w = r_{i_1} \cdot \cdots \cdot r_{i_k}$ is a reduced expression for $w \in W$, then $\delta(x, y) = w$ if and only if x and y can be joined by a gallery of type $i_1 \cdots i_k$.

We say that a building is *thick*, if every panel belongs to at least three chambers. It is called *thin*, if every panel is common to exactly two chambers. It is easy to prove that thin buildings are nothing other than Coxeter complexes, with a W-distance $\delta(x, y) = x^{-1}y$.

Apartments. A map $\alpha \colon W \to \Delta$ is said to be an *isometry* if it preserves the W–distance δ. In other words,

$$\delta(\alpha(x), \alpha(y)) = x^{-1}y$$

for all $x, y \in W$. An *apartment* is an isometric image $\alpha[W]$ of W in Δ. Apartments exist by Theorem 3.6 in [Ron]. Moreover, by Corollary 3.7 in [Ron] any two chambers of Δ lie in a common apartment.

Retractions. It is easy to prove (see [Ron], p. 32) that any isometry $\alpha \colon W \to \Delta$ is uniquely determined by its image $A = \alpha[W]$ together with the chamber $c = \alpha(1)$.

Now fix any apartment A and chamber $c \in A$. Let $A = \alpha[W]$ with $c = \alpha(1)$. We define a *retraction*

$$\rho_{c,A} \colon \Delta \to A$$

of Δ *onto* A *with center* c as

$$\rho_{c,A}(x) = \alpha\Big(\delta(c,x)\Big).$$

The following properties of retractions easily follow from the definition (see also [Ti1], Theorem 3.3 and Lemma 3.6).

Fact 10. The retraction $\rho_{c,A}$ is an idempotent morphism of Δ onto A. Moreover,

(a) If $d \in A$ and $x \in \Delta$, then $\rho_{c,A}$ maps any gallery connecting d and x onto a gallery in A, connecting d and $\rho_{c,A}(x)$.
(b) $\rho_{c,A}$ maps any geodesic gallery connecting c and x onto a geodesic gallery connecting c and $\rho_{c,A}(x)$.

7.3. Geometric realizations of W–matroids. The following theorem is the most important result of the present paper.

Theorem 4. *Let* Δ *be a thick building of type* W *and* $\alpha\colon W \to \Delta$ *an isometry of* W *into* Δ. *We identify the apartment* $\alpha[W]$ *of* Δ *with* W *via* α, *so by abuse of language we assume* $W \subset \Delta$. *Fix some chamber* $x \in \Delta$. *Then for any* $u, w \in W$ *we have*

$$w^{-1}\rho_{w,W}(x) \geq w^{-1}\rho_{u,W}(x).$$

In particular, the map

$$\mu(w) = \rho_{w,W}(x)$$

is a W*–matroid.*

Proof. Let Γ be a geodesic gallery connecting w and x. By Fact 10(b) the retraction $\rho_{w,W}$ maps Γ onto a geodesic gallery Γ' stretched from w to $\rho_{w,W}(x)$. But by Fact 10(a) $\rho_{u,W}$ maps Γ onto a gallery Γ'' connecting $\rho_{u,W}(w) = w$ and $\rho_{u,W}(x)$. The geometric interpretation of the Bruhat ordering (Lemma 4) yields immediately that

$$w^{-1}\rho_{w,W}(x) \geq w^{-1}\rho_{u,W}(x). \quad \square$$

We say that a triple

$$(\Delta, \alpha\colon W \to \Delta, x)$$

is a *geometric realization* of a W–matroid $\mu\colon W \to W$, if

$$\mu(w) = \rho_{w,W}(x)$$

for all $w \in W$.

Question 1. Which W–matroids have a geometric realization?

Certainly, this is a very difficult problem. In particular, it includes the question about existence of buildings of the given type W. In the special case when $W = \mathrm{Sym}_n$ is the symmetric group on $n \geq 4$ letters, buildings of type W are flag complexes of projective spaces [Ti1], so our problem includes, as a partial case, the question of representability of matroids in the classical meaning of these words ([GS2], [Aig], [Wel]).

8. Tits Systems and Thin Schubert Cells

8.1. Tits systems.

Definition of Tits systems. We say that a group G contains a *Tits system* (B, N), if the following conditions hold.

(1) B and N are subgroups of G and $G = BNB$.
(2) $H = B \cap N$ is a normal subgroup of N and $W = N/H$ is generated by a set $R = \{r_i,\, i \in I\}$ of involutions.
(3) $r_i BwB \subseteq BwB \cup Br_iwB$ for any $w \in W$, $r_i \in R$.
(4) For each $r_i \in R$, we have $r_iBr_i \neq B$.

Such expressions as r_iBwB have unambiguous meaning, since r_i and w are cosets of H and thus subsets of G; if $\bar{r}_i$ and $\bar{w}$ are representatives of r_i and w in N, we have $r_iBwB = \bar{r}_iB\bar{w}B$.

The group W is called the *Weyl group* of the Tits system. It is well known that W is a Coxeter group and $R = \{r_i,\, i \in I\}$ is the distinguished set of generators for W. A Tits system (B, N) is said to be of *spherical type*, if W is finite.

If G is the group of k–points of a reductive algebraic group defined over a field k, B is a minimal k–parabolic subgroup in G and $H < B$ is a maximal k–split torus of G, then $(B, N_G(H))$ is a Tits system in G of spherical type ([Ti1], Theorem 5.2). Examples of Tits systems with infinite Weyl groups are provided by reductive groups over local fields [BrT] and by Kac-Moody groups [MoT].

Buildings associated with Tits systems. One of the main properties of Tits systems is the *Bruhat decomposition* (see [Ron], Lemma 5.1):

$$G = \bigsqcup_{w \in W} BwB$$

is a disjoint union. In particular, each $g \in G$ uniquely determines an element $w \in W$ such that $g \in BwB$.

Fact 11 (*J.Tits*[Ti1], *Theorem 3.2.6,* [Ron], *Theorem 5.3*). Every Tits system (B, N) defines a building Δ of type W, the chambers being left cosets of B, with i–adjacency given by

$$gB \sim_i hB \iff g^{-1}h \in B \cup Br_iB.$$

Moreover,

$$\delta(gB, hB) = w \iff g^{-1}h \in BwB$$

is the W–distance function on this building. The subgroup B is the stabilizer of the chamber B in the action G on $\Delta = G/B$ by the left multiplication and N stabilizes the apartment $\{wB,\, w \in W\}$.

8.2. Schubert cells and retractions.

Schubert cells. The image in G/B of a double coset $wBw^{-1}gB$ with respect to a pair of subgroups wBw^{-1} and B is called a *Schubert cell.*

Lemma 5. *Let $g \in G$. Any Schubert cell $wBw^{-1}gB/B$ can be written in the form*

$$wBw^{-1}gB/B = wBw^{-1}uB/B$$

for some $u \in W$. The element u is uniquely determined by the cell. In particular, there is a decomposition

$$G/B = \bigsqcup_{u \in W} wBw^{-1}uB/B$$

of G/B into a disjoint union of Schubert cells with representatives $u \in W$.

Proof. By the Bruhat decomposition an element $w^{-1}g$ can be presented in the form

$$w^{-1}g = b_1vb_2$$

for some $b_1, b_2 \in B$ and $v \in W$. The element v is uniquely determined by g. Set $u = wv$, then

$$w^{-1}g = b_1w^{-1}ub_2$$

and

$$g = wb_1w^{-1}ub_2 \in wBw^{-1}uB,$$

where $u \in W$ is uniquely determined by g. But it means that we have a disjoint union

$$G = \bigsqcup_{u \in W} wBw^{-1}uB,$$

and also

$$G/B = \bigsqcup_{u \in W} wBw^{-1}uB/B.$$

□

It follows immediately from the definition of Δ that the map $w \mapsto wB$ is an isometry of W onto the apartment $\{wB,\, w \in W\}$ of Δ. We can identify W with this apartment.

Lemma 6. *Under these assumptions, if $g \in wBw^{-1}uB$, then*

$$u = \rho_{w,W}(gB),$$

where $\rho_{w,W}$ is the retraction of Δ onto W with the center w.

Proof. The left multiplication by w is an isometry of W onto W which sends 1 to w, so the definition of the retraction $\rho_{w,W}$ takes the form

$$\rho_{w,W}(x) = w\delta(w, x) \text{ for } x \in \Delta.$$

Let now $x = gB$. By definition,

$$\delta(wB, gB) = v \iff w^{-1}g \in BvB.$$

But $g \in wBw^{-1}uB$, so $w^{-1}g \in Bw^{-1}uB$ and we should take $v = w^{-1}u$. But then

$$\rho_{w,W}(gB) = w\delta(wB, gB) = wv = w \cdot w^{-1}u = u. \quad \square$$

Thin Shubert cells and W–matroids. Under the assumptions of the previous paragraph, set

$$\mu_g(w) = \rho_{w,W}(gB).$$

By the previous lemma

$$g \in wBw^{-1}\mu_g(w)B,$$

and, taking the intersection over all $w \in W$, we have

$$g \in \bigcap_{w \in W} wBw^{-1}\mu_g(w)B.$$

The image K of this intersection in G/B is called a *thin Schubert cell*. Obviously thin Schubert cells form a partition of G/B, generated by all partitions of G/B into Schubert cells. It follows from Lemma 6, that the function $\mu_g \colon W \to W$ does not depend on the choice of an element $gB \in K$, whence can be denoted by μ_K.

The following theorem is an immediate corollary of Lemma 4 and Lemma 6.

Theorem 5. *Let G be a group with a Tits system (B, N) and $W = N/N \cap B$ its Weyl group. Consider a thin Schubert cell*

$$K = \bigcap_{w \in W} wBw^{-1}\mu_K(w)B/B$$

in G/B associated with a function $\mu_K \colon W \to W$. Then μ_K is a W–matroid.

9. WP–matroids

In this section we generalize our results from W–matroids to WP–matroids.

9.1. Residues and parabolic subgroups.

Residues. Let C be a chamber complex over I and $J \subseteq I$. The relation

> *x and y can be connected by a gallery of type $i_1 \cdots i_m$ with all $i_k \in J$, $1 \le k \le m$,*

is clearly an equivalence relation on C. Its classes are called *J–residues* ([Ron]) or *faces of type J* ([Ti1]). Notice that faces of type $\{i\}$ are just i–panels and faces of type $\varnothing$ are chambers. Given residues χ and ψ of types J and K respectively we say that ψ is a face of χ if $\psi \supset \chi$ and $K \supset J$.

Clearly any morphism of chamber systems over I sends faces of type J to faces of type J.

If now W is a Coxeter group, which we identify with its Coxeter complex, then it is easy to see that faces of type J are left cosets wP with respect to a parabolic subgroup $P = \langle r_i,\, i \in J\rangle$. So the set of all faces of type J can be identified with the factor set $W^P = W/P$.

The Bruhat ordering on W^P. The following fact makes it clear how to define the Bruhat ordering on W^P.

Fact 12 (*V. Deodhar* [Deo], *Lemma 2.1*). (a) Any coset $\alpha \in W^P$ contains the minimal, with respect to the Bruhat ordering, element w_α.

(b) Let α, $\beta \in W^P$ be two cosets and $a \in \alpha$, $b \in \beta$ any representatives of α and β, correspondingly. If $a \le b$, then $w_\alpha \le w_\beta$.

Following [Deo], we introduce a partial ordering $\le$ on W^P by putting $\alpha \le \beta$ if $w_\alpha \le w_\beta$. In view of Fact 12 this is equivalent to the condition that $a \le b$ for some representatives $a \in \alpha$, $b \in \beta$.

WP–matroids. We define a *WP–matroid* as a convex map

$$\mu\colon W \to W^P.$$

It means that μ satisfies the inequality

$$w^{-1}\mu(u) \leq w^{-1}\mu(w)$$

for all $u, w \in W$.

One can define for subsets in W^P the maximality and minimality conditions by analogy with these notions for subsets in W (Section 6). If $M = \mu[W]$ is the image in W^P of a WP–matroid μ, then for any $w \in W$, M contains a w–maximal element $\mu(w)$. Hence for finite Coxeter groups, where the maximality condition for W^P coincides with the minimality condition, our definition of a WP–matroid is equivalent to the definition of a WP–matroid in the sense of [GS1] (the work [GS1] defines WP–matroids as subsets in W^P satisfying the minimality condition). Thus our notion of a WP–matroid includes as partial cases the notion of a matroid.

Notice that W acts on the set $\mathcal{M}_P(W)$ of all WP–matroids on W by the following rule: if $u \in W$, then

$$(u \cdot \mu)\colon w \mapsto u^{-1}\mu(uw).$$

9.2. Geometric interpretation of WP–matroids.

Geometric interpretation of the Bruhat ordering on W^P. Let C be a chamber system and π, σ two faces of C. We say that a gallery $\Gamma = (c_0, \dots, c_k)$ connects π and σ, if π is a face of c_0 and σ is a face of c_k.

The following lemma is an easy consequence of Lemma 4 and the definition of the ordering of W^P.

Lemma 7. *Let $w \in W$, $P = \langle r_i,\, i \in J\rangle$ a parabolic subgroup, $\omega = wP$ the face of w of type J,*

$$\Gamma = (x_0, \dots, x_k), \quad w = x_0,$$

a geodesic gallery and

$$\Gamma' = (y_0, \dots, y_k), \quad w = y_0,$$

a gallery of the same type $i_1 \cdots i_k$, connecting ω to faces $\chi \in W^P$ and $\psi \in W^P$, correspondingly. Then

$$w^{-1}\psi \leq w^{-1}\chi.$$

WP–matroids and retractions of buildings. The following result is an immediate generalization of Theorem 4.

Theorem 6. *Let Δ be a thick building of type W and $\alpha\colon W \to \Delta$ an isometry of W into Δ. We identify the apartment $\alpha[W]$ of Δ with W via α. Fix some face $\chi \in \Delta$ of type J. Then for any u, $w \in W$ we have*

$$w^{-1}\rho_{u,W}(\chi) \leq w^{-1}\rho_{w,W}(\chi),$$

where $\geq$ is understood as the Bruhat ordering on the set W^P, $P = \langle r_i,\, i \in J\rangle$, of all faces of W of type J. In particular, the function

$$\mu\colon W \to W^P,$$
$$\mu\colon w \mapsto \rho_{w,W}(\chi)$$

is a WP–matroid.

Geometric realizations of WP–matroids. In the conditions of Theorem 6 a triple

$$(\Delta, \alpha\colon W \to \Delta, \chi)$$

is called a *geometric realization* of a WP–matroid μ.

Question 2. Which WP–matroids have a geometric realization?

Schubert cells on flag spaces G/G_J. Save notations of Section 8. If $J \subset I$, let $P = P_J = \langle r_i,\, i \in J\rangle$ be a parabolic subgroup in W generated by J, $W^P = W/P_J$ and $G_J = BP_JB$. As easily follows from the definition of a Tits system, G_J is a subgroup of G. It is called a *parabolic subgroup of type* J. We shall identify the building Δ of a Tits system (B, N) with the factor set G/B. Then G_J is the stabilizer in G of the face of type J of the chamber B (see [Ron], Theorem 5.4). The group G is transitive on the set of all faces of type J in Δ, so we can identify the set of all faces of type J with G/G_J and call it the *flag space of type* J for the group G.

One can easily prove that for any $w \in W$ there is a decomposition

$$G = \bigsqcup_{\alpha \in W^P} wBw^{-1}\alpha G_J$$

into a disjoint union of double cosets with respect to subgroups $wBw^{-1}\alpha G_J$. Taking the natural projection of G onto G/G_J, we consider this partition as a decomposition of G/G_J:

$$G/G_J = \bigsqcup_{\alpha \in W^P} wBw^{-1}\alpha G_J/G_J.$$

Lemma 8. *Save the notations above. If $g \in wBw^{-1}\alpha G_J$, then*

$$\alpha = \rho_{w,W}(gG_J)P_J,$$

where $\rho_{w,W}$ is the retraction of Δ onto W with the center w.

Thin Shubert cells and WP–matroids. Under the assumptions of the previous lemma, we denote

$$\mu_g(w) = \rho_{w,W}(gG_J)P_J.$$

By the previous lemma

$$g \in wBw^{-1}\mu_g(w)G_J,$$

and, taking the intersection over all $w \in W$, we have

$$gG_J \in K = \bigcap_{w \in W} wBw^{-1}\mu_g(w)G_J/G_J.$$

The set K is called a *thin Schubert cell* on the flag space G/G_J. Obviously thin Schubert cells form a partition of G/G_J generated by all partitions of G/G_J into Schubert cells. Moreover, the partition into thin Schubert cells is invariant under the action of the Weyl group W: if

$$K = \bigcap_{w \in W} wBw^{-1}\mu_g(w)G_J/G_J$$

is a thin Schubert cell and $u \in W$, then

$$uK = \bigcap_{w \in W} uwB(uw)^{-1}\mu_g(uw)G_J/G_J$$

is another thin Schubert cell.

It follows from Lemma 8 that the function $\mu_g \colon W \to W^P$ does not depend on the choice of a coset $gG_J \in K$ and thus can be denoted by μ_K.

The following theorem is a corollary of Lemma 7 and Lemma 8.

Theorem 7. *Let G be a group with a Tits system (B, N), $W = N/N \cap B$ its Weyl group and $R = \{r_i,\ i \in I\}$ the distinguished set of generators for W. Let $J \subseteq I$, $P = \langle r_i,\ i \in J\rangle$, $W^P = W/P$ and $G_J = BPB$. Consider a thin Schubert cell*

$$K = \bigcap_{w \in W} wBw^{-1}\mu_K(w)G_J/G_J$$

on the flag space G/G_J, associated with a function $\mu_K\colon W \to W^P$. Then μ_K is a WP–matroid.

This theorem generalizes Theorem 2 of Section 8.3 in [GS2].

9.3. Submatroids and Zariski closures of thin Schubert cells.

Submatroids. Let W be a Coxeter group, P a parabolic subgroup in W and

$$\mu, \lambda\colon W \to W/P$$

two WP–matroids. We say that μ is a *submatroid* of λ and write $\mu \leq \lambda$, if

$$w^{-1}\mu(w) \leq w^{-1}\lambda(w)$$

for all $w \in W$.

Theorem 8. *If W is a finite Coxeter group, P a parabolic subgroup in W and μ, λ are WP–matroids, then $\mu \leq \lambda$ if and only if*

$$\mu[W] \subseteq \lambda[W].$$

Proof. implies. Recall that the multiplication by the longest element $w_0 \in W$ reverses the Bruhat ordering. It means that wP is the maximal element of W with respect to the ordering $\leq^{w_0 w}$. So if $wP \in \mu[W]$, then

$$w_P = \mu(w_0 w) \leq^{w_0 w} \lambda(w_0 w)$$

and, $wP = \lambda(w_0 w) \in \lambda[W]$.

The implication $\Longleftarrow$ is obvious. $\square$

Zariski closure of thin Schubert cells. In this section we assume that G is a simple algebraic group over an algebraically closed field, B a Borel subgroup in G, H a maximal torus in B and $N = N_G(H)$. It is well known that (B, N) is a Tits system in G of spherical type and that if G_J a parabolic subgroup in G, then G/G_J is a complete algebraic variety [Hu1].

Fact 13 (*Borel - Tits* [BoT]). If, under these assumptions, BwG_J is a Schubert cell in G/G_J, then its Zariski closure has a form

$$\overline{BwG_J} = \bigcup_{u\in W^P, u\leq w} BuG_J.$$

Theorem 9. *Let K_μ be a thin Schubert cell on G/G_J corresponding to a WP–matroid μ. Then*

$$\overline{K_\mu} \subseteq \bigcup_{\lambda\leq\mu} K_\lambda.$$

Proof. It is almost an immediate consequence of Fact 13. We need only to notice that

$$\begin{aligned}\overline{wBw^{-1}uG_J} &= \bigcup_{v\in W^P, w^{-1}v\leq w^{-1}u} wBw^{-1}vG_J \\ &= \bigcup_{v\in W^P, v\leq^w u} wBw^{-1}vG_J.\end{aligned}$$

So if

$$K_\mu = \bigcap_{w\in W} wBw^{-1}\mu(w)G_J,$$

then

$$\begin{aligned}\overline{K_\mu} &\subseteq \bigcap_{w\in W} \overline{wBw^{-1}\mu(w)G_J} \\ &= \bigcap_{w\in W}\left(\bigcup_{v\in W^P, v\leq^w \mu(w)} wBw^{-1}vG_J\right) \\ &= \bigcup_{\lambda\leq\mu}\left(\bigcap_{w\in W} wBw^{-1}\lambda(w)G_J\right) \\ &= \bigcup_{\lambda\leq\mu} K_\lambda,\end{aligned}$$

where λ is a WP–matroid, possibly corresponding to an empty thin Schubert cell. The theorem follows at once. □

Notice that there are examples showing that in general

$$\overline{K_\mu} \neq \bigcup_{\lambda\leq\mu} K_\lambda$$

(see [GS2]).

Closures of orbits of a maximal torus. Obviously, since thin Schubert cells are H–invariant, their Zariski closures are also H–invariant. Every

H–orbit HgG_J/G_J can be assigned, in view of Theorem 7, a WP–matroid μ.

The following theorem is an immediate corollary of Theorem 9.

Theorem 10. *Let μ be a WP–matroid of a H–orbit HgG_J/G_J on a flag variety G/G_J. If λ is a WP–matroid corresponding to an orbit Hg_1G_J/G_J in the Zariski closure $\overline{HgG_J/G_J}$, then $\lambda \leq \mu$.*

If $K = \mathbb{C}$ is the field of complex numbers and G is a simple algebraic group over $\mathbb{C}$, then the results of Sections 4–8 in the work by I. M. Gelfand and V. V. Serganova [GS2] can be summed up as follows:

Theorem 11. *Let HgG_J/G_J be an H–orbit on the flag space G/G_J corresponding to a WP–matroid μ. Then there is one-to-one correspondence between H–orbits in the Zariski closure $\overline{HgG_J/G_J}$ and submatroids of μ.*

This theorems gives, in a partial case of realizations of WP–matroids in buildings of simple algebraic groups over the field of complex numbers, the positive answer to the following question on geometric realizations of matroids.

Question 3. Assume that WP–matroid μ has a geometric realization in a thick building Δ of type W. If $\lambda \leq \mu$ is another WP–matroid, is it true that λ also has a realization in Δ?

9.4. Realizations of chamber matroids. Let Δ be a connected chamber system over I (it means that any two chambers can be connected by a gallery) and $A \subseteq \Delta$ a subsystem in Δ. An idempotent morphism $\rho\colon \Delta \to A$ is called a *retraction of* Δ *onto* A *with the center* $c \in A$, if for any chamber $x \in \Delta$, ρ maps any geodesic gallery stretched in Δ from the chamber c to x onto a geodesic gallery in A stretched from $c = \rho(c)$ to $\rho(x)$. If A is a thin chamber system, then clearly a geodesic gallery stretched between two chambers in A is uniquely determined by its type, therefore the retraction $\rho\colon \Delta \to A$ is uniquely determined by its center c. We say that A is a *flat* in Δ, if A is a thin chamber complex and for any $c \in A$ there exists a (unique) retraction of Δ on A with the center c. We denote it by $\rho_{c,A}$.

The following theorem is almost obvious.

Theorem 12. *If A is a flat in a connected chamber system Δ and $x \in \Delta$, then the map*

$$\mu(c) = \rho_{c,A}(x)$$

is a matroid on A.

We say in this situation that a triple

$$(\Delta, A \hookrightarrow \Delta, x)$$

is a realization of a matroid μ.

Question 4. Which chamber matroids have a realization?

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