The $\alpha$-labeling number of comets is 2

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Abstract
We investigate the claim that for every tree $T$ (with $m$ edges), there exists an $\alpha$-labeling of $T$, or else there exists a graph $H_T$ with an $\alpha$-labeling such that $H_T$ can be decomposed into two edge-disjoint copies of $T$. We prove that the above claim is true for comets $C_{m, 2}$. This is particularly noteworthy since comets $C_{m, 2}$ are known to have arbitrarily large $\alpha$-deficits.

1 Introduction
Given a graph $G$, an injective function $f : V(G) \to \mathbb{N}$ is called a vertex labeling, or a vertex numbering of $G$. Such a function $f$ on a graph $G$ with $m$ edges is known as a graceful-labeling if $f$ is an injection from $V(G)$ to the set $\{0, 1, \ldots, m\}$ such that the values $|f(x) - f(y)|$ for all $m$ pairs of adjacent vertices $x, y$ are distinct. A labeling $f$ is bipartite if there exists an integer $\lambda$ so that for each edge $xy$ either $f(x) \leq \lambda < f(y)$ or $f(y) \leq \lambda < f(x)$. A labeling $f$ is an $\alpha$-labeling if it is graceful and bipartite.

Clearly, if $G$ has an $\alpha$-labeling, then $G$ must be bipartite. Suppose $G$ is bipartite with $m$ edges and degree-sequence $d_1, d_2, \ldots, d_n$. Wu [1] showed that the necessary condition for $G$ having an $\alpha$-labeling is

$$\gcd(d_1, d_2, \ldots, d_n, m) \mid \binom{m}{2}.$$  

*Hunter Snevily passed away on November 11, 2013 after his long struggle with Parkinson’s disease. We have lost a good friend and colleague. He will be greatly missed and fondly remembered.
The following theorem is a classical result on $\alpha$-labeling of graphs.

**Theorem 1** (Rosa [3]). Let $G$ be a graph with $m$ edges, and let $G$ have an $\alpha$-labeling. Then the complete graph $K_{2pm+1}$ can be decomposed into isomorphic copies of $G$, where $p$ is an arbitrary positive integer.

Snevily [8] introduced the following graph parameter motivated by Rosa’s result:

A bipartite graph $G$ with $m$ edges eventually has an $\alpha$-labeling if there exists a graph $H$ with $t \cdot m$ edges (where $t$ is a positive integer), such that $H$ has an $\alpha$-labeling and can be decomposed into edge-disjoint copies of $G$. Such a graph $H$ is called the host graph of $G$.

Suppose $G$ is a bipartite graph that eventually has an $\alpha$-labeling; then the $\alpha$-labeling number of $G$, denoted $G_\alpha$, is defined as follows:

$$G_\alpha = \min \{ t : \exists \text{ a host graph } H \text{ such that } |E(H)| = t \cdot m \}.$$  

Snevily [8] conjectured that for every bipartite graph $G$, $G_\alpha < \infty$, which was later proved by El-Zanati, Fu and Shiue [2]. There are no known examples of a graph $G$ with $G_\alpha > 2$ (See Gallian [3]). Snevily also conjectured that for a tree $T$ with $m$ edges, $T_\alpha \leq m$. Shiue and Fu [6] proved that $\alpha$-labeling number for a tree with $m$ edges and radius $r$ is at most $\lceil r/2 \rceil m$. They also prove that a tree with $m$ edges and radius $r$ decomposes $K_t$ for some $t \leq (r + 1)m^2 + 1$.

In this paper, we conjecture the following:

**Conjecture 1.** For any tree $T$,

$$T_\alpha \leq 2.$$  

For a tree $T$, the $\alpha$-deficit $\alpha_{df}(T)$ equals $m - \alpha(T)$, where $\alpha(T)$ is defined as the maximum number of distinct edge labels over all bipartite labelings of $T$.

**Observation 1** ([8]). Let $G = (X, Y)$ be a bipartite graph with $m$ edges and consider the graph $rG$ consisting of $r$ disjoint copies of $G$. Suppose there exists a labeling function

$$h : V(rG) \to \{0, 1, 2, \ldots, rm\}$$

such that

(i) the labels assigned to the vertices in any single copy of $G$ (in $rG$) are distinct,

(ii) if $(x, y) \in E(rG)$, then the value $|h(x) - h(y)|$ is assigned to the edge $(x, y)$, and no other edge in $E(rG)$,
there exists some real number $\lambda_h$ such that if $G_i = (X_i, Y_i)$ is some copy of $G$ in $rG$ then
\[
\max \{ h(x) : x \in X_i \} \leq \lambda_h < \min \{ h(y) : y \in Y_i \},
\]
or else
\[
\max \{ h(y) : y \in Y_i \} \leq \lambda_h < \min \{ h(x) : x \in X_i \}.
\]

Let
\[
S = \{ x : x \in V(rG) \text{ and } h(x) \leq \lambda_h \}
\]
and
\[
T = \{ y : y \in V(rG) \text{ and } h(y) > \lambda_h \}.
\]
Clearly, $S$ and $T$ are independent sets. Now we can take the labeled version of $rG$ and create a new graph $H$ by identifying vertices (from different copies of $G$) with the same label. Hence $H$ is a bipartite graph with $|E(H)| = rm$, and that $H$ has $\alpha$-labeling. Clearly $H$ is a host graph of $G$.

2 \hspace{1cm} \alpha\text{-labeling number of comets}

The comet $C_{m,k}$ is obtained from the star $K_{1,m}$ by replacing each edge in $K_{1,m}$ with a path of length $k$. Rosa and Širáň \[5\] showed that for every $m \geq 1$,
\[
\alpha_{\text{def}}(C_{m,2}) = \lfloor m/3 \rfloor,
\]
which implies that $(C_{m,2})_\alpha \geq 2$ for $m \geq 3$.

Let $C_{m,j}$ be a comet-like tree with a central vertex of degree $m$, and each neighbor of the central vertex is attached to $j$ pendant vertices where $j \geq 1$. Here, $C_{m,2} = C_{m,1}$.

2.1 \hspace{1cm} Construction for $(C'_{m,j})_\alpha$ where $m \geq 3$ and $j \geq 1$

Comet $C'_{m,j}$ has $1 + m + mj$ vertices and $m + mj$ edges. We construct a graph $2C'_{m,j}$ with $2m(j + 1)$ edges that has an $\alpha$-labeling and can be decomposed into two edge-disjoint copies isomorphic to $C'_{m,j}$.

We start with two disjoint copies $C_1$ and $C_2$ of $C'_{m,j}$ and then we utilize Observation \[6\]. Note that there are three types of vertices in $C'_{m,j}$: one central vertex of degree $m$, $m$ vertices of degree $j + 1$, and $mj$ pendant vertices.

Let the central vertices in $C_1$ and $C_2$ be $x_0$ and $y_0$, respectively. Let the degree-$(j + 1)$ vertices in $C_1$ and $C_2$ be $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_m$, respectively.
respectively. Let in $C_1$, the pendant vertices attached to $x_i$ with $1 \leq i \leq m$ be

$$x_{m+(i-1)j+1}, x_{m+(i-1)j+2}, \ldots, x_{m+(i-1)j+j}$$

and in $C_2$, the pendant vertices attached to $y_i$ with $1 \leq i \leq m$ be

$$y_{m+(i-1)j+1}, y_{m+(i-1)j+2}, \ldots, y_{m+(i-1)j+j}.$$ 

We define a labeling function

$$h : \{x_0, x_1, \ldots, x_{m+j}, y_0, y_1, \ldots, y_{m+j}\} \rightarrow \{0, 1, 2, \ldots, 2m+2mj\}.$$ 

Label $x_0$ and $y_0$ as 0 and $2m+j$, respectively. The vertices $x_1, x_2, \ldots, x_m$ in $C_1$ and $y_1, y_2, \ldots, y_m$ in $C_2$ share $m$ labels in common, which are $2m+j, 2m+2, 2m+3, \ldots, 2m+2m$.

Now, for the $k$-th pendant vertex attached to $x_i$ and $y_i$ for $i = 1, 2, \ldots, m$, set

(i) $m$ odd:

$$h(x_{m+(i-1)j+k}) = (2i-1) + (k-1)m, \text{ and}$$

$$h(y_{m+(i-1)j+k}) = h(x_{m+(i-1)j+k}) + mj.$$ 

respectively. For example, $2C'_3, 2$ looks as follows:

(ii) $m$ even: $h(x_{m+(i-1)j+k})$ equals

$$\begin{align*}
&\begin{cases}
  m + (2t - 1) + (t - 1)2m, & \text{if } k = 2t; \\
  m + mj + (2i - 1) + (t - 1)2m, & \text{if } k = 2t - 1 \text{ and } j \text{ even}; \\
  mj + (2i - 1) + (t - 1)2m, & \text{if } k = 2t - 1 \text{ and } j \text{ odd}.
\end{cases}
\end{align*}$$

and

$$h(y_{m+(i-1)j+k}) = h(x_{m+(i-1)j+k}) - m.$$ 

Example ($2C'_4, 2$):
Example (2C'_{4,3}): 

Lemma 1. Both $C_1$ and $C_2$ have distinct vertex labels.

Proof. Define 

$$ g(i, k, r) = \begin{cases} 
  h(x_{m+(i-1)j+k}), & \text{if } r = 1; \\
  h(y_{m+(i-1)j+k}), & \text{if } r = 2.
\end{cases} $$

Now we consider the following cases:

(i) $(m \text{ odd})$: Here, $1 \leq g(i, k, 1) \leq mj + m - 1$ and $mj + 1 \leq g(i, k, 2) \leq 2mj + m - 1$. The sequence 

$$ g(1, 1, 1), \quad g(2, 1, 1), \quad \cdots \quad g(m, 1, 1), $$
$$ g(1, 3, 1), \quad g(2, 3, 1), \quad \cdots \quad g(m, 3, 1), $$
$$ \vdots \quad \vdots \quad \vdots \quad \vdots $$
$$ g(1, 2t - 1, 1), \quad g(2, 2t - 1, 1), \quad \cdots \quad g(m, 2t - 1, 1). $$

is a strictly increasing sequence of $m \lfloor j/2 \rfloor$ odd numbers since 

(a) $g(1, 1, 1) = (2 - 1) + (1 - 1)m = 1$,

(b) For $i = 1, 2, \ldots, m - 1$ and $1 \leq t \leq \lfloor j/2 \rfloor$,

$$ g(i + 1, 2t - 1, 1) = g(i, 2t - 1, 1) + 2, $$

(c) For $t = 1, 2, \ldots, \lfloor j/2 \rfloor - 1$,

$$ g(1, 2t + 1, 1) = (2 - 1) + (2t + 1 - 1)m $$
$$ = (2m - 1) + (2t - 1 - 1)m + 2 $$
$$ = g(m, 2t - 1, 1) + 2. $$

And the sequence 

$$ g(1, 2, 1), \quad g(2, 2, 1), \quad \cdots \quad g(m, 2, 1), $$
$$ g(1, 4, 1), \quad g(2, 4, 1), \quad \cdots \quad g(m, 4, 1), $$
$$ \vdots \quad \vdots \quad \vdots \quad \vdots $$
$$ g(1, 2t, 1), \quad g(2, 2t, 1), \quad \cdots \quad g(m, 2t, 1), $$

is a strictly increasing sequence of $m \lceil j/2 \rceil$ even numbers since
(a) \( g(1, 2, 1) = (2 - 1) + (2 - 1)m = m + 1, \)
(b) For \( i = 1, 2, \ldots, m - 1 \) and \( 1 \leq t \leq \lfloor j/2 \rfloor, \)
\[
g(i + 1, 2t, 1) = g(i, 2t, 1) + 2,
\]
(c) For \( t = 1, 2, \ldots, \lfloor j/2 \rfloor - 1, \)
\[
g(1, 2t + 2, 1) = (2 - 1) + (2t + 1)m
\]
\[
= (2m - 1) + (2t - 1)m + 2
\]
\[
= g(m, 2t, 1) + 2.
\]
Together, the \( m \lfloor j/2 \rfloor + m \lfloor j/2 \rfloor = mj \) distinct numbers label the pendant vertices of \( C_1. \) Since \( h(y_{m+(i-1)j+k}) = h(x_{m+(i-1)j+k}) + mj, \)
\( C_2 \) also has distinct vertex-labels for the pendant vertices.

(ii) \( (m \text{ even, } j \text{ even}): \) Consider the sequence
\[
\begin{align*}
g(1, 2, 2), & \quad g(2, 2, 2), \quad \ldots \quad g(m, 2, 2), \\
g(1, 4, 2), & \quad g(2, 4, 2), \quad \ldots \quad g(m, 4, 2), \\
& \quad \vdots \\
g(1, j, 2), & \quad g(2, j, 2), \quad \ldots \quad g(m, j, 2), \\
g(1, 1, 2), & \quad g(2, 1, 2), \quad \ldots \quad g(m, 1, 2), \\
g(1, 3, 2), & \quad g(2, 3, 2), \quad \ldots \quad g(m, 3, 2), \\
& \quad \vdots \\
g(1, j - 1, 2), & \quad g(2, j - 1, 2), \quad \ldots \quad g(m, j - 1, 2),
\end{align*}
\]
which is a strictly increasing sequence of \( mj \) odd numbers since
(a) \( g(1, 2, 2) = 1, \)
(b) For \( i = 1, 2, \ldots, m - 1 \) and \( 1 \leq t \leq j/2, \)
\[
g(i + 1, 2t, 2) = (2i + 1) + (t - 1)2m
\]
\[
= (2i - 1) + (t - 1)2m + 2
\]
\[
= g(i, 2t, 2) + 2,
\]
(c) For \( t = 1, 2, \ldots, (j - 2)/2, \)
\[
g(1, 2t + 2, 2) = (2 - 1) + (t + 1 - 1)2m
\]
\[
= (2m - 1) + (t - 1)2m + 2
\]
\[
= g(m, 2t, 2) + 2,
\]
(d) \( g(1, 1, 2) = mj + (2 - 1) + (1 - 1)2m = (2m - 1) + (j/2 - 1)2m + 2 = g(m, j, 2) + 2. \)
(e) For \( i = 1, 2, \ldots, m - 1 \) and \( 1 \leq t \leq j/2 \),
\[
g(i + 1, 2t - 1, 2) = mj + (2i + 1) + (t - 1)2m \\
= mj + (2i - 1) + (t - 1)2m + 2 \\
= g(i, 2t - 1, 2) + 2,\]

(f) For \( t = 1, 2, \ldots, (j - 2)/2 \),
\[
g(1, 2t + 1, 2) = mj + (2 - 1) + (t + 1 - 1)2m \\
= mj + (2m - 1) + (t - 1)2m + 2 \\
= g(m, 2t - 1, 2) + 2.
\]

Hence, \( C_2 \) has distinct vertex-labeling and so does \( C_1 \).

(iii) \((m \text{ even, } j \text{ odd})\): This may be demonstrated with the same argument as in the previous case, but using the sequence

\[
\begin{align*}
g(1, 2, 2), & \quad g(2, 2, 2), \quad \cdots \quad g(m, 2, 2), \\
g(1, 4, 2), & \quad g(2, 4, 2), \quad \cdots \quad g(m, 4, 2), \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \\
g(1, j - 1, 2), & \quad g(2, j - 1, 2), \quad \cdots \quad g(m, j - 1, 2), \\
g(1, 1, 2), & \quad g(2, 1, 2), \quad \cdots \quad g(m, 1, 2), \\
g(1, 3, 2), & \quad g(2, 3, 2), \quad \cdots \quad g(m, 3, 2), \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \\
g(1, j, 2), & \quad g(2, j, 2), \quad \cdots \quad g(m, j, 2).
\end{align*}
\]

Lemma 2. \( 2C_{m,j}' \) has distinct edge-labeling, that is, each edge \((x, y) \in E(2C_{m,j}')\) has a distinct value of \(|h(x) - h(y)|\) in \( \{1, 2, \ldots, 2m + 2mj\} \).

**Proof.** By construction, for \( i = 1, 2, \ldots, m \),
\[
|h(x_0) - h(x_i)| = |0 - (m + 2mj + i)| = m + 2mj + i, \\
|h(y_0) - h(y_i)| = |(m + 2mj) - (m + 2mj + i)| = i.
\]

We need to show that the remaining \( 2mj \) edges, each of which is connected to a pendant vertex, have distinct labels using
\[
m + 1, m + 2, \ldots, m + 2mj.
\]

Define
\[
f(i, k, r) = \begin{cases} 
    h(x_i) - h(x_{m+(i-1)j+k}) & \text{if } r = 1; \\
    h(y_i) - h(y_{m+(i-1)j+k}) & \text{if } r = 2.
\end{cases}
\]
Note that for positive integers $1 \leq i \leq m$, $1 \leq k \leq j$, and $1 \leq r \leq 2$, there are exactly $2mj$ input combinations for $f(i, k, r)$. Now we consider the following cases:

(i) $(m$ odd$)$: Consider the following sequence:

\[
\begin{align*}
&f(m, j, 2), \quad f(m - 1, j, 2), \quad \cdots \quad f(1, j, 2), \\
&f(m, j - 1, 2), \quad f(m - 1, j - 1, 2), \quad \cdots \quad f(1, j - 1, 2), \\
&f(m, j - 2, 2), \quad f(m - 1, j - 2, 2), \quad \cdots \quad f(1, j - 2, 2), \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \\
&f(m, 1, 2), \quad f(m - 1, 1, 2), \quad \cdots \quad f(1, 1, 2),
\end{align*}
\]

We claim that the $mj$ numbers in the sequence are $m + 1, m + 2, \ldots, m + mj$, which can be observed from the following:

(a) The first number,
\[
\begin{align*}
f(m, j, 2) &= h(y_m) - h(y_{m+(m-1)j+j}) \\
&= (2mj + 2m) - (mj + (2m - 1) + (j - 1)m) \\
&= m + 1.
\end{align*}
\]

(b) For $i = 1, 2, \ldots, m - 1$ and $1 \leq k \leq j$,
\[
\begin{align*}
f(i, k, 2) &= h(y_i) - h(y_{m+(i-1)j+k}) \\
&= (2mj + 2m + i) - (mj + (2i - 1) + (k - 1)m) \\
&= (2mj + 2m + i + 1) - 1 \\
&\quad - (mj + (2i + 1) + (k - 1)m) + 2 \\
&= f(i + 1, k, 2) + 1.
\end{align*}
\]

(c) For $k = 1, 2, \ldots, j - 1$,
\[
\begin{align*}
f(m, k, 2) &= h(y_m) - h(y_{m+(2m-1)j+k}) \\
&= (2mj + 2m) - (mj + (2m - 1) + (k - 1)m) \\
&= (2mj + m + 1) + m - 1 \\
&\quad - (mj + (2 - 1) + (k + 1 - 1)m) - m + 2 \\
&= f(1, k + 1, 2) + 1.
\end{align*}
\]

(d) The last number,
\[
\begin{align*}
f(1, 1, 2) &= h(y_1) - h(y_{m+(1-1)j+1}) \\
&= (2mj + m + 1) - (mj + (2 - 1) + (1 - 1)m) \\
&= m + mj.
\end{align*}
\]
Similarly, the $mj$ numbers in the sequence
\[
\begin{align*}
  f(m, j, 1), & \quad f(m - 1, j, 1), \quad \cdots \quad f(1, j, 1), \\
  f(m, j - 1, 1), & \quad f(m - 1, j - 1, 1), \quad \cdots \quad f(1, j - 1, 1), \\
  f(m, j - 2, 1), & \quad f(m - 1, j - 2, 1), \quad \cdots \quad f(1, j - 2, 1), \\
  \vdots & \quad \vdots \quad \cdots \quad \vdots \\
  f(m, 1, 1), & \quad f(m - 1, 1, 1), \quad \cdots \quad f(1, 1, 1),
\end{align*}
\]
represent the numbers
\[m + mj + 1, m + mj + 2, \ldots, m + 2mj,\]
since
\[
\begin{align*}
  f(m, j, 1) & = m + mj + 1, \\
  f(i, k, 1) & = f(i + 1, k, 1) + 1 \text{ for } i = 1, 2, \ldots, m - 1, \\
  f(m, k, 1) & = f(1, k + 1, 1) + 1 \text{ for } k = 1, 2, \ldots, j - 1, \\
  f(1, 1, 1) & = m + 2mj.
\end{align*}
\]

(ii) (m even, j even):

Consider the following sequence:
\[
\begin{align*}
  f(m, j - 1, 1), & \quad f(m - 1, j - 1, 1), \quad \cdots \quad f(1, j - 1, 1), \\
  f(m, j - 2, 1), & \quad f(m - 1, j - 2, 1), \quad \cdots \quad f(1, j - 2, 1), \\
  f(m, j - 3, 1), & \quad f(m - 1, j - 3, 1), \quad \cdots \quad f(1, j - 3, 1), \\
  f(m, j - 3, 2), & \quad f(m - 1, j - 3, 2), \quad \cdots \quad f(1, j - 3, 2), \\
  \vdots & \quad \vdots \quad \cdots \quad \vdots \\
  f(m, 1, 1), & \quad f(m - 1, 1, 1), \quad \cdots \quad f(1, 1, 1), \\
  f(m, 1, 2), & \quad f(m - 1, 1, 2), \quad \cdots \quad f(1, 1, 2).
\end{align*}
\]
We claim that the $mj$ numbers in the sequence are $m + 1, m + 2, \ldots, m + mj$, which can be observed from the following:

(a) The first number,
\[
f(m, j - 1, 1) = h(x_m) - h(x_{m + (m - 1)j + (j - 1)}) = (2mj + 2m) - (m + mj + (2m - 1) + (j/2 - 1)2m) = m + 1.
\]

(b) For $i = 1, 2, \ldots, m - 1$ and $1 \leq t \leq j/2$,
\[
f(i, 2t - 1, 1) = h(x_i) - h(x_{(i-1)j + (2t-1)}) \\
= (2mj + m + i) - (m + mj + (2i - 1) + (t - 1)2m) \\
= (2mj + m + i + 1 - 1 \\
= -(m + mj + (2i + 1) - 1) + (t - 1)2m) + 2 \\
= f(i + 1, 2t - 1, 1) + 1.
\]
Similarly, for \( i = 1, 2, \ldots, m - 1 \) and \( 1 \leq t \leq j/2 \),
\[
f(i, 2t - 1, 2) = f(i + 1, 2t - 1, 2) + 1.
\]

(c) For \( t = 1, 2, \ldots, j/2 \),
\[
f(m, 2t - 1, 2) = h(y_m) - h(y_{m + (m-1)t + 2t-1})
\]
\[
= (2mj + 2m) - (mj + (2m - 1) + (t - 1)2m)
\]
\[
= (2mj + m + 1)
\]
\[
-(m + mj + (2 - 1) + (t - 1)2m) + 1
\]
\[
= h(x_1) - h(x_{m + (m-1)t + 2t-1}) + 1
\]
\[
= f(1, 2t - 1, 1) + 1.
\]

(d) For \( t = 1, 2, \ldots, (j - 2)/2 \),
\[
f(m, 2t - 1, 1) = h(x_m) - h(x_{m + (m-1)t + 2t-1})
\]
\[
= (2mj + 2m) - (m + mj + (2m - 1) + (t - 1)2m)
\]
\[
= (2mj + m + 1) + m - 1 - m
\]
\[
-(mj + (2 - 1) + ((t + 1) - 1)2m) + 2
\]
\[
= h(y_1) - h(y_{m + (m-1)t + 2t+1}) + 1
\]
\[
= f(1, 2t + 1, 2) + 1.
\]

(e) The last number,
\[
f(1, 1, 2) = x_1 - x_{m + (m-1)t + 1}
\]
\[
= (2mj + m + 1) - (mj + (2 - 1) + (1 - 1)2m) = m + mj.
\]

Similarly, the \( mj \) numbers in the sequence
\[
f(m, j, 1), \quad f(m - 1, j, 1), \quad \cdots \quad f(1, j, 1),
\]
\[
f(m, j, 2), \quad f(m - 1, j, 2), \quad \cdots \quad f(1, j, 2),
\]
\[
f(m, j - 2, 1), \quad f(m - 1, j - 2, 1), \quad \cdots \quad f(1, j - 2, 1),
\]
\[
f(m, j - 2, 2), \quad f(m - 1, j - 2, 2), \quad \cdots \quad f(1, j - 2, 2),
\]
\[
\vdots \quad \vdots \quad \cdots \quad \vdots
\]
\[
f(m, 2, 1), \quad f(m - 1, 2, 1), \quad \cdots \quad f(1, 2, 1),
\]
\[
f(m, 2, 2), \quad f(m - 1, 2, 2), \quad \cdots \quad f(1, 2, 2).
\]

represent the numbers
\[
m + mj + 1, m + mj + 2, \ldots, m + 2mj.
\]
since

\[
\begin{align*}
    f(m, j, 1) &= m + mj + 1, \\
    f(i, 2t, 1) &= f(i + 1, 2t, 1) + 1 \text{ for } i = 1, 2, \ldots, m - 1, \\
    f(i, 2t, 2) &= f(i + 1, 2t, 2) + 1 \text{ for } i = 1, 2, \ldots, m - 1, \\
    f(m, 2t, 2) &= f(1, 2t, 1) + 1 \text{ for } t = 1, 2, \ldots, j/2, \\
    f(m, 2t, 1) &= f(1, 2t + 2, 2) + 1 \text{ for } t = 1, 2, \ldots, (j - 2)/2, \\
    f(1, 2, 2) &= m + 2mj.
\end{align*}
\]

\text{(iii) (m even, j odd):}

It can be shown as in the previous case that the \(m(j + 1)\) numbers in the sequence

\[
\begin{align*}
    f(m, j, 1), & \quad f(m - 1, j, 1), & \cdots & \quad f(1, j, 1), \\
    f(m, j, 2), & \quad f(m - 1, j, 2), & \cdots & \quad f(1, j, 2), \\
    f(m, j - 2, 1), & \quad f(m - 1, j - 2, 1), & \cdots & \quad f(1, j - 2, 1), \\
    f(m, j - 2, 2), & \quad f(m - 1, j - 2, 2), & \cdots & \quad f(1, j - 2, 2), \\
    \vdots & & \vdots & \cdots & \vdots \\
    f(m, 1, 1), & \quad f(m - 1, 1, 1), & \cdots & \quad f(1, 1, 1), \\
    f(m, 1, 2), & \quad f(m - 1, 1, 2), & \cdots & \quad f(1, 1, 2).
\end{align*}
\]

represent the numbers

\[
m + 1, m + 2, \ldots, 2m + mj,
\]

since

\[
\begin{align*}
    f(m, j, 1) &= m + 1, \\
    f(i, 2t - 1, 1) &= f(i + 1, 2t - 1, 1) + 1 \\
    & \quad \text{for } i = 1, 2, \ldots, m - 1 \text{ and } 1 \leq t \leq (j + 1)/2, \\
    f(i, 2t - 1, 2) &= f(i + 1, 2t - 1, 2) + 1 \\
    & \quad \text{for } i = 1, 2, \ldots, m - 1 \text{ and } 1 \leq t \leq (j + 1)/2, \\
    f(m, 2t - 1, 2) &= f(1, 2t + 1, 1) + 1 \text{ for } t = 1, 2, \ldots, (j - 1)/2, \\
    f(m, 2t - 1, 1) &= f(1, 2t + 1, 2) + 1 \text{ for } t = 1, 2, \ldots, (j - 3)/2, \\
    f(1, 2, 2) &= 2m + 2mj.
\end{align*}
\]
And, the \( m(j - 1) \) numbers in the sequence

\[
\begin{align*}
&f(m, j - 1, 1), \quad f(m - 1, j - 1, 1), \quad \cdots \quad f(1, j - 1, 1), \\
&f(m, j - 1, 2), \quad f(m - 1, j - 1, 2), \quad \cdots \quad f(1, j - 1, 2), \\
&f(m, j - 3, 1), \quad f(m - 1, j - 3, 1), \quad \cdots \quad f(1, j - 3, 1), \\
&f(m, j - 3, 2), \quad f(m - 1, j - 3, 2), \quad \cdots \quad f(1, j - 3, 2), \\
&\vdots \quad \vdots \quad \cdots \quad \vdots \\
&f(m, 2, 1), \quad f(m - 1, 2, 1), \quad \cdots \quad f(1, 2, 1), \\
&f(m, 2, 2), \quad f(m - 1, 2, 2), \quad \cdots \quad f(1, 2, 2).
\end{align*}
\]

represent the numbers

\[
2m + mj + 1, 2m + mj + 2, \ldots, m + 2mj,
\]
since

\[
\begin{align*}
f(m, j - 1, 1) &= 2m + mj + 1, \\
f(i, 2t, 1) &= f(i + 1, 2t, 1) + 1 \\
&\quad \text{for } i = 1, 2, \ldots, m - 1 \text{ and } 1 \leq t \leq (j - 1)/2, \\
f(i, 2t, 2) &= f(i + 1, 2t, 2) + 1 \\
&\quad \text{for } i = 1, 2, \ldots, m - 1 \text{ and } 1 \leq t \leq (j - 1)/2, \\
f(m, 2t, 2) &= f(1, 2t, 1) + 1 \text{ for } t = 1, 2, \ldots, (j - 1)/2, \\
f(m, 2t, 1) &= f(1, 2t + 2, 2) + 1 \text{ for } t = 1, 2, \ldots, (j - 3)/2, \\
f(1, 2, 2) &= m + 2mj.
\end{align*}
\]

\[\square\]

**Theorem 2.** For \( m \geq 3 \), \( \left( C'_{m,j}\right)_\alpha = 2 \) where \( j \geq 1 \).

**Proof.** The proof follows from Lemmas \( \Box \) and \( \Box \), and Observation \( \Box \). \[\square\]

## 3 Trees with \( \alpha \)-deficits

In this section, we have relied on the results of Brinkmann et al. in \( \Box \).

**Conjecture 2.** If \( \Delta_T = 2k + 1 \), then \( \alpha_{def}(T) \leq k \).

**Conjecture 3.** For all \( k \geq 1 \) and for all \( 2 \leq j \leq 2k \),

\[
\alpha_{def}(C'_{2k+1,j}) = k.
\]

**Lemma 3.** For \( k \geq 1 \) and \( 2 \leq j \leq 2k \),

\[
\alpha_{def} \left( C'_{2k+1,j} \right) \leq k.
\]
Proof. Consider the graph $C_{m,j}'$ with $m = 2k + 1$. Let the vertices be
\[ x_0, x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+mj} \]
where $x_0$ is the central vertex with degree $m$, each of the vertices $x_1, x_2, \ldots, x_m$ has degree $j+1$, and $x_{m+1}, x_{m+2}, \ldots, x_{m+mj}$ are the pendant vertices. Consider the vertex labeling $h$ with $h(x_0) = 0$, $h(x_i) = mj + i$ for $i = 1, 2, \ldots, m$ and
\[
h(x_{m+(i-1)j+r}) = \begin{cases} (2i - 1) + (r - 1)m, & \text{for } 1 \leq i \leq m, 1 \leq r \leq j - 1; \\ (2i - 1) + (j - 1)m, & \text{for } 1 \leq i \leq m - k. \end{cases}
\]
Similar to the $m$-odd case of Lemma 1, the vertices $x_0, x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+mj}$ have distinct labels from $0, 1, 2, \ldots, m + mj$. Similar to the $m$-odd case of Lemma 2, all edges have distinct labels except that the labels for the $k$ edges $(x_i, x_{m+(i-1)j+j})$ with $i = m-k+1, m-k+2, \ldots, m$ are missing. \qed

**Proposition 1.** For $k \geq 1$ and $2 \leq j \leq 2k$,
\[
\alpha_{\text{def}}(C_{2k+1,j}') > 0.
\]

**Proof.** Let $G = C_{m,j}'$ where $m = 2k + 1$ with vertices
\[ x_0, x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+mj} \]
where $x_0$ is the central vertex with degree $m$ and $x_{m+1}, x_{m+2}, \ldots, x_{m+mj}$ are the pendant vertices. Assume that $G$ has an $\alpha$-labeling $\ell$. Then, the sum of all edge-labels,
\[
S = \sum_{i=1}^{m+mj} i = (m + mj)(m + mj + 1)/2 \equiv 0 \pmod{m}.
\]

By Remark B1 of Brinkmann et al. \cite{ref}, let the vertices $x_i$ for $i = 1, 2, \ldots, m$ be labeled with $mj + i$, respectively. The remaining numbers $0, 1, 2, \ldots, mj$ label $x_0$ and the pendant vertices. For any choice of $\ell(x_0) \in \{0, 1, 2, \ldots, mj\}$, we have
\[
S_1 = \sum_{i=1}^{m} (\ell(x_i) - \ell(x_0)) = \sum_{i=1}^{m} (\ell(x_i)) - \sum_{i=1}^{m} (\ell(x_0)) = m^2j + m(m+1)/2 - ml(x_0) \equiv 0 \pmod{m}.
\]
Since $\ell$ is an $\alpha$-labeling, for $i = 1, 2, \ldots, m$ and $t = 1, 2, \ldots, j$, the pendant vertices $x_{m+(i-1)j+t}$ are labeled in such a way that

$$S_2 = \sum_{i=1}^{m} \sum_{t=1}^{j} (\ell(x_i) - \ell(x_{m+(i-1)j+t}))$$

$$= j \sum_{i=1}^{m} \ell(x_i) - \sum_{i=1}^{m} \sum_{t=1}^{j} (\ell(x_{m+(i-1)j+t}))$$

$$\equiv 0 \pmod{m}, \quad \text{(since } S = S_1 + S_2 \text{ and } S, S_1 \equiv 0 \pmod{m})$$

implying

$$\sum_{i=1}^{m} \sum_{t=1}^{j} (\ell(x_{m+(i-1)j+t})) \equiv 0 \pmod{m},$$

which holds only if $\ell(x_0)$ is chosen from the $mj + 1$ labels $0, 1, \ldots, mj$ in such a way that

$$\ell(x_0) \equiv 0 \pmod{m}.$$

Suppose $\ell(x_0) = 0$. Then the edge-labels of $(x_0, x_i)$ for $i = 1, 2, \ldots, m$ are

$$mj + 1, mj + 2, \ldots, mj + m.$$

Since the labels less than $mj + 1$ must still be used, we may determine the following locations for vertex labels in order:

1 can only label a vertex attached to $x_1$, adding $mj$ to the set of edge-labels,

2 can only label a vertex attached to $x_1$, adding $mj - 1$ to the set of edge-labels,

... ,

j can only label a vertex attached to $x_1$, adding $mj - (j - 1)$ to the set of edge-labels.

All the pendant vertices attached to $x_1$ are labeled and $j + 1$ cannot be used to label any pendant vertex attached to $x_i$ for $i = 2, 3, \ldots, m$. Hence $\ell(x_0) \neq 0$.

Let $\ell(x_0) = tm$ with $1 \leq t \leq j$. Then the edge-labels of $(x_0, x_i)$ for $i = 1, 2, \ldots, m$ are

$$mj - mt + 1, mj - mt + 2, \ldots, mj - mt + m.$$

As above, we may determine the locations of certain vertex labels:

The only way to add edge-label $m + mj$ is to use 0 to label a vertex attached to $x_m$.

The only way to add edge-label $m + mj - 1$ is to use 1 to label a vertex attached to $x_m$. 

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The only way to add edge-label $m + mj - (j - 1)$ is to use $j - 1$ to label a vertex attached to $x_m$.

All the pendant vertices attached to $x_m$ are labeled and the labels used are $0, 1, 2, \ldots, j - 1$. But the only way to add the edge-label $m + mj - j$ is to use $r \in \{0, 1, 2, \ldots, j - 1\}$ to label a pendant vertex attached to $x_{m-j+r}$ so that

$$\ell(x_{m-j+r}) - r = mj + (m - j + r) - r = m + mj - j,$$

which is impossible. Hence, we have a contradiction to our assumption that $G$ has an $\alpha$-labeling.

\[\square\]

4 Concluding remarks

In this paper, we have given an example of constructing a graph with a graceful, bipartite labeling which can be decomposed into two isomorphic edge-disjoint trees consisting of a root node of degree $m$, each of whose neighbours is connected to $j$ ($j \geq 1$) leaves.

This result is a special case of the conjecture that for every tree $T$, two copies of $T$ can be packed into a graph with a graceful, bipartite labeling. The result remotely connects to the graceful tree conjecture which states that all trees are graceful. We have also explored the extent to which a bipartite labeling falls short of gracefulness.

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References


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