The α-labeling number of comets is 2

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Abstract

We investigate the claim that for every tree T (with m edges), there exists an α-labeling of T, or else there exists a graph H_T with an α-labeling such that H_T can be decomposed into two edge-disjoint copies of T. We prove that the above claim is true for comets $\mathcal{C}_{m,2}$. This is particularly noteworthy since comets $\mathcal{C}_{m,2}$ are known to have arbitrarily large α-deficits.

1 Introduction

Given a graph G, an injective function $f : V(G) \to \mathbb{N}$ is called a *vertex labeling*, or a *vertex numbering* of G. Such a function f on a graph G with m edges is known as a *graceful-labeling* if f is an injection from $V(G)$ to the set $\{0, 1, \ldots, m\}$ such that the values $|f(x)-f(y)|$ for all m pairs of adjacent vertices x, y are distinct. A labeling f is *bipartite* if there exists an integer λ so that for each edge xy either $f(x) \leqslant \lambda < f(y)$ or $f(y) \leqslant \lambda < f(x)$. A labeling f is an *α-labeling* if it is graceful and bipartite.

Clearly, if G has an α-labeling, then G must be bipartite. Suppose G is bipartite with m edges and degree-sequence $d_1, d_2, \ldots, d_n$. Wu [9] showed that the necessary condition for G having an α-labeling is

$$gcd(d_1, d_2, \ldots, d_n, m) \mid \binom{m}{2}.$$

*Hunter Snevily passed away on November 11, 2013 after his long struggle with Parkinson's disease. We have lost a good friend and colleague. He will be greatly missed and fondly remembered.

The following theorem is a classical result on α-labeling of graphs.

Theorem 1 (Rosa [4]). Let G be a graph with m edges, and let G have an α-labeling. Then the complete graph K_{2pm+1} can be decomposed into isomorphic copies of G, where p is an arbitrary positive integer.

Snevily [8] introduced the following graph parameter motivated by Rosa's result:

A bipartite graph G with m edges *eventually has an α-labeling* if there exists a graph H with $t \cdot m$ edges (where t is a positive integer), such that H has an α-labeling and can be decomposed into edge-disjoint copies of G. Such a graph H is called the *host* graph of G.

Suppose G is a bipartite graph that eventually has an α-labeling; then the *α-labeling number* of G, denoted G_α is defined as follows:

$$G_\alpha = \min \{t : \exists \text{ a host graph } H \text{ such that } |E(H)| = t \cdot m\}.$$

Snevily [8] conjectured that for every bipartite graph G, $G_\alpha < \infty$, which was later proved by El-Zanati, Fu and Shiue [2]. There are no known examples of a graph G with $G_\alpha > 2$ (See Gallian [3]). Snevily also conjectured that for a tree T with m edges, $T_\alpha \leqslant m$. Shiue and Fu [6] proved that α-labeling number for a tree with m edges and radius r is at most $\lceil r/2 \rceil m$. They also prove that a tree with m edges and radius r decomposes K_t for some $t \leqslant (r+1)m^2 + 1$.

In this paper, we conjecture the following:

Conjecture 1. For any tree T,

$$T_\alpha \leqslant 2.$$

For a tree T, the α-deficit $\alpha_{def}(T)$ equals $m - \alpha(T)$, where $\alpha(T)$ is defined as the maximum number of distinct edge labels over all bipartite labelings of T.

Observation 1 ([8]). Let $G = (X, Y)$ be a bipartite graph with m edges and consider the graph rG consisting of r disjoint copies of G. Suppose there exists a labeling function

$$h : V(rG) \to \{0, 1, 2, \ldots, rm\}$$

such that

(i) the labels assigned to the vertices in any single copy of G (in rG) are distinct,

(ii) if $(x, y) \in E(rG)$, then the value $|h(x) - h(y)|$ is assigned to the edge (x, y), and no other edge in $E(rG)$, (*iii*) there exists some real number λ_h such that if $G_i = (X_i, Y_i)$ is some copy of G in rG then

$$\max\{h(x) : x \in X_i\} \leqslant \lambda_h < \min\{h(y) : y \in Y_i\},$$

or else

$$\max\{h(y) : y \in Y_i\} \leqslant \lambda_h < \min\{h(x) : x \in X_i\}.$$

Let

$$S = \{x : x \in V(rG) \text{ and } h(x) \leqslant \lambda_h\}$$

and

$$T = \{y : y \in V(rG) \text{ and } h(y) > \lambda_h\}.$$

Clearly, S and T are independent sets. Now we can take the labeled version of rG and create a new graph H by identifying vertices (from different copies of G) with the same label. Hence H is a bipartite graph with $|E(H)| = rm$, and that H has α-labeling. Clearly H is a host graph of G.

2 α-labeling number of comets

The *comet* $\mathcal{C}_{m,k}$ is obtained from the star $K_{1,m}$ by replacing each edge in $K_{1,m}$ with a path of length k. Rosa and Širáň [5] showed that for every $m \geqslant 1$,

$$\alpha_{def}(\mathcal{C}_{m,2}) = \lfloor m/3 \rfloor,$$

which implies that $(\mathcal{C}_{m,2})_\alpha \geqslant 2$ for $m \geqslant 3$.

Let $\mathcal{C}'_{m,j}$ be a comet-like tree with a central vertex of degree m, and each neighbour of the central vertex is attached to j pendant vertices where $j \geqslant 1$. Here, $\mathcal{C}_{m,2} = \mathcal{C}'_{m,1}$.

2.1 Construction for $(\mathcal{C}'_{m,j})_\alpha$ where $m \geqslant 3$ and $j \geqslant 1$

Comet $\mathcal{C}'_{m,j}$ has $1 + m + mj$ vertices and $m + mj$ edges. We construct a graph $2\mathcal{C}'_{m,j}$ with $2m(j+1)$ edges that has an α-labeling and can be decomposed into two edge-disjoint copies isomorphic to $\mathcal{C}'_{m,j}$.

We start with two disjoint copies C_1 and C_2 of $\mathcal{C}'_{m,j}$ and then we utilize Observation 1. Note that there are three types of vertices in $\mathcal{C}'_{m,j}$: one central vertex of degree m, m vertices of degree $j+1$, and mj pendant vertices.

Let the central vertices in C_1 and C_2 be x_0 and y_0, respectively. Let the degree-$(j+1)$ vertices in C_1 and C_2 be $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_m$, respectively. Let in C_1, the pendant vertices attached to x_i with $1 \leqslant i \leqslant m$ be

$$x_{m+(i-1)j+1}, x_{m+(i-1)j+2}, \ldots, x_{m+(i-1)j+j}$$

and in C_2, the pendant vertices attached to y_i with $1 \leqslant i \leqslant m$ be

$$y_{m+(i-1)j+1}, y_{m+(i-1)j+2}, \ldots, y_{m+(i-1)j+j}.$$

We define a labeling function

$$h : \{x_0, x_1, \ldots, x_{m+mj}, y_0, y_1, \ldots, y_{m+mj}\} \to \{0, 1, 2, \ldots, 2m + 2mj\}.$$

Label x_0 and y_0 as 0 and $2mj+m$, respectively. The vertices $x_1, x_2, \ldots, x_m$ in C_1 and $y_1, y_2, \ldots, y_m$ in C_2 share m labels in common, which are

$$2mj + m + 1, 2mj + m + 2, 2mj + m + 3, \ldots, 2mj + 2m,$$

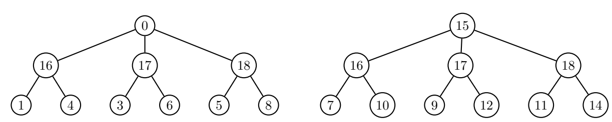
in the same order from left to right for the indices $i = 1, 2, \ldots, m$.

Now, for the k-th pendant vertex attached to x_i and y_i for $i = 1, 2, \ldots, m$, set

(i) m odd:

$$\begin{aligned} h(x_{m+(i-1)j+k}) &= (2i-1) + (k-1)m, \text{ and} \\ h(y_{m+(i-1)j+k}) &= h(x_{m+(i-1)j+k}) + mj. \end{aligned}$$

respectively. For example, $2\mathcal{C}'_{3,2}$ looks as follows:



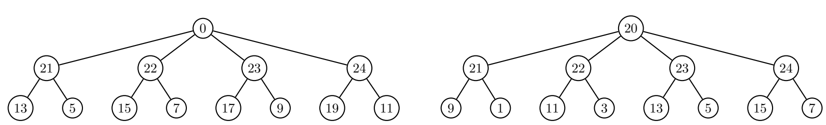
(ii) m even: $h(x_{m+(i-1)j+k})$ equals

$$\begin{cases} m + (2i-1) + (t-1)2m, & \text{if } k = 2t; \\ m + mj + (2i-1) + (t-1)2m, & \text{if } k = 2t-1 \text{ and } j \text{ even}; \\ mj + (2i-1) + (t-1)2m, & \text{if } k = 2t-1 \text{ and } j \text{ odd}. \end{cases}$$

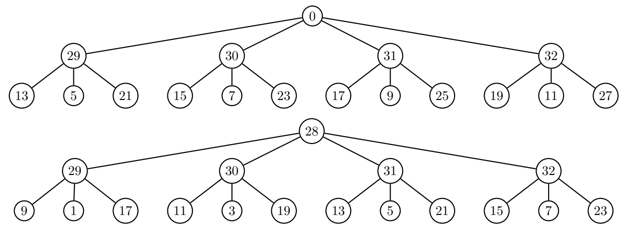
and

$$h(y_{m+(i-1)j+k}) = h(x_{m+(i-1)j+k}) - m.$$

Example ($2\mathcal{C}'_{4,2}$):



Example ($2\mathcal{C}'_{4,3}$):



Lemma 1. Both C_1 and C_2 have distinct vertex labels.

Proof. Define

$$g(i,k,r) \quad = \quad \begin{cases} h(x_{m+(i-1)j+k}), & \text{if } r=1; \\ h(y_{m+(i-1)j+k}), & \text{if } r=2. \end{cases}$$

Now we consider the following cases:

(i) (m odd): Here, $1 \leqslant g(i,k,1) \leqslant mj+m-1$ and $mj+1 \leqslant g(i,k,2) \leqslant 2mj+m-1$. The sequence

$$\begin{array}{cccc} g(1,1,1), & g(2,1,1), & \cdots & g(m,1,1), \\ g(1,3,1), & g(2,3,1), & \cdots & g(m,3,1), \\ \vdots & \vdots & \vdots & \vdots \\ g(1,2t-1,1), & g(2,2t-1,1), & \cdots & g(m,2t-1,1). \end{array}$$

is a strictly increasing sequence of $m\lceil j/2 \rceil$ odd numbers since

(a) $g(1,1,1) = (2-1)+(1-1)m = 1$,

(b) For $i = 1,2,\ldots,m-1$ and $1 \leqslant t \leqslant \lceil j/2 \rceil$,

$$g(i+1,2t-1,1) = g(i,2t-1,1)+2,$$

(c) For $t = 1,2,\ldots,\lceil j/2 \rceil - 1$,

$$\begin{aligned} g(1,2t+1,1) &= (2-1)+(2t+1-1)m \\ &= (2m-1)+(2t-1-1)m+2 \\ &= g(m,2t-1,1)+2. \end{aligned}$$

And the sequence

$$\begin{array}{cccc} g(1,2,1), & g(2,2,1), & \cdots & g(m,2,1), \\ g(1,4,1), & g(2,4,1), & \cdots & g(m,4,1), \\ \vdots & \vdots & \vdots & \vdots \\ g(1,2t,1), & g(2,2t,1), & \cdots & g(m,2t,1), \end{array}$$

is a strictly increasing sequence of $m\lfloor j/2 \rfloor$ even numbers since

(a) $g(1,2,1) = (2-1) + (2-1)m = m+1$,

(b) For $i = 1,2,\ldots,m-1$ and $1 \leqslant t \leqslant \lfloor j/2 \rfloor$,

$$g(i+1,2t,1) = g(i,2t,1) + 2,$$

(c) For $t = 1,2,\ldots,\lfloor j/2 \rfloor - 1$,

$$\begin{aligned} g(1,2t+2,1) &= (2-1) + (2t+1)m \\ &= (2m-1) + (2t-1)m + 2 \\ &= g(m,2t,1) + 2. \end{aligned}$$

Together, the $m\lceil j/2 \rceil + m\lfloor j/2 \rfloor = mj$ distinct numbers label the pendant vertices of C_1. Since $h(y_{m+(i-1)j+k}) = h(x_{m+(i-1)j+k}) + mj$, C_2 also has distinct vertex-labels for the pendant vertices.

(ii) (m even, j even): Consider the sequence

$$\begin{array}{cccc} g(1,2,2), & g(2,2,2), & \cdots & g(m,2,2), \\ g(1,4,2), & g(2,4,2), & \cdots & g(m,4,2), \\ \vdots & \vdots & \vdots & \vdots \\ g(1,j,2), & g(2,j,2), & \cdots & g(m,j,2), \\ g(1,1,2), & g(2,1,2), & \cdots & g(m,1,2), \\ g(1,3,2), & g(2,3,2), & \cdots & g(m,3,2), \\ \vdots & \vdots & \vdots & \vdots \\ g(1,j-1,2), & g(2,j-1,2), & \cdots & g(m,j-1,2), \end{array}$$

which is a strictly increasing sequence of mj odd numbers since

(a) $g(1,2,2) = 1$,

(b) For $i = 1,2,\ldots,m-1$ and $1 \leqslant t \leqslant j/2$,

$$\begin{aligned} g(i+1,2t,2) &= (2i+1) + (t-1)2m \\ &= (2i-1) + (t-1)2m + 2 \\ &= g(i,2t,2) + 2, \end{aligned}$$

(c) For $t = 1,2,\ldots,(j-2)/2$,

$$\begin{aligned} g(1,2t+2,2) &= (2-1) + (t+1-1)2m \\ &= (2m-1) + (t-1)2m + 2 \\ &= g(m,2t,2) + 2, \end{aligned}$$

(d) $g(1,1,2) = mj + (2-1) + (1-1)2m = (2m-1) + (j/2-1)2m + 2 = g(m,j,2) + 2$,

(e) For $i = 1, 2, \ldots, m-1$ and $1 \leqslant t \leqslant j/2$,

$$\begin{aligned} g(i+1, 2t-1, 2) &= mj + (2i+1) + (t-1)2m \\ &= mj + (2i-1) + (t-1)2m + 2 \\ &= g(i, 2t-1, 2) + 2, \end{aligned}$$

(f) For $t = 1, 2, \ldots, (j-2)/2$,

$$\begin{aligned} g(1, 2t+1, 2) &= mj + (2-1) + (t+1-1)2m \\ &= mj + (2m-1) + (t-1)2m + 2 \\ &= g(m, 2t-1, 2) + 2. \end{aligned}$$

Hence, C_2 has distinct vertex-labeling and so does C_1.

(iii) (m even, j odd): This may be demonstrated with the same argument as in the previous case, but using the sequence

$$\begin{array}{cccc} g(1,2,2), & g(2,2,2), & \cdots & g(m,2,2), \\ g(1,4,2), & g(2,4,2), & \cdots & g(m,4,2), \\ \vdots & \vdots & \vdots & \vdots \\ g(1,j-1,2), & g(2,j-1,2), & \cdots & g(m,j-1,2), \\ g(1,1,2), & g(2,1,2), & \cdots & g(m,1,2), \\ g(1,3,2), & g(2,3,2), & \cdots & g(m,3,2), \\ \vdots & \vdots & \vdots & \vdots \\ g(1,j,2), & g(2,j,2), & \cdots & g(m,j,2). \end{array}$$

□

Lemma 2. $2\mathcal{C}'_{m,j}$ has distinct edge-labeling, that is, each edge $(x, y) \in E(2\mathcal{C}'_{m,j})$ has a distinct value of $|h(x) - h(y)|$ in $\{1, 2, \ldots, 2m + 2mj\}$.

Proof. By construction, for $i = 1, 2, \ldots, m$,

$$\begin{aligned} |h(x_0) - h(x_i)| &= |0 - (m + 2mj + i)| = m + 2mj + i, \\ |h(y_0) - h(y_i)| &= |(m + 2mj) - (m + 2mj + i)| = i. \end{aligned}$$

We need to show that the remaining $2mj$ edges, each of which is connected to a pendant vertex, have distinct labels using

$$m+1, m+2, \ldots, m+2mj.$$

Define

$$f(i,k,r) = \begin{cases} h(x_i) - h(x_{m+(i-1)j+k}), & \text{if } r = 1; \\ h(y_i) - h(y_{m+(i-1)j+k}), & \text{if } r = 2. \end{cases}$$

Note that for positive integers $1 \leqslant i \leqslant m$, $1 \leqslant k \leqslant j$, and $1 \leqslant r \leqslant 2$, there are exactly $2mj$ input combinations for $f(i,k,r)$. Now we consider the following cases:

(i) (m odd): Consider the following sequence:

$$\begin{array}{cccc} f(m,j,2), & f(m-1,j,2), & \cdots & f(1,j,2), \\ f(m,j-1,2), & f(m-1,j-1,2), & \cdots & f(1,j-1,2), \\ f(m,j-2,2), & f(m-1,j-2,2), & \cdots & f(1,j-2,2), \\ \vdots & \vdots & \vdots & \vdots \\ f(m,1,2), & f(m-1,1,2), & \cdots & f(1,1,2), \end{array}$$

We claim that the mj numbers in the sequence are $m+1, m+2, \ldots, m+mj$, which can be observed from the following:

(a) The first number,

$$\begin{aligned} f(m,j,2) &= h(y_m) - h(y_{m+(m-1)j+j}) \\ &= (2mj+2m) - (mj + (2m-1) + (j-1)m) \\ &= m+1. \end{aligned}$$

(b) For $i = 1, 2, \ldots, m-1$ and $1 \leqslant k \leqslant j$,

$$\begin{aligned} f(i,k,2) &= h(y_i) - h(y_{m+(i-1)j+k}) \\ &= (2mj+2m+i) - (mj + (2i-1) + (k-1)m) \\ &= (2mj+2m+i+1) - 1 \\ &\quad -(mj + (2i+1) + (k-1)m) + 2 \\ &= f(i+1,k,2) + 1. \end{aligned}$$

(c) For $k = 1, 2, \ldots, j-1$,

$$\begin{aligned} f(m,k,2) &= h(y_m) - h(y_{m+(2m-1)j+k}) \\ &= (2mj+2m) - (mj + (2m-1) + (k-1)m) \\ &= (2mj+m+1) + m - 1 \\ &\quad -(mj + (2-1) + (k+1-1)m) - m + 2 \\ &= f(1,k+1,2) + 1. \end{aligned}$$

(d) The last number,

$$\begin{aligned} f(1,1,2) &= h(y_1) - h(y_{m+(1-1)j+1}) \\ &= (2mj+m+1) - (mj + (2-1) + (1-1)m) \\ &= m+mj. \end{aligned}$$

Similarly, the mj numbers in the sequence

$$\begin{array}{cccc} f(m,j,1), & f(m-1,j,1), & \cdots & f(1,j,1), \\ f(m,j-1,1), & f(m-1,j-1,1), & \cdots & f(1,j-1,1), \\ f(m,j-2,1), & f(m-1,j-2,1), & \cdots & f(1,j-2,1), \\ \vdots & \vdots & \vdots & \vdots \\ f(m,1,1), & f(m-1,1,1), & \cdots & f(1,1,1), \end{array}$$

represent the numbers

$$m+mj+1, m+mj+2, \ldots, m+2mj,$$

since

$$\begin{array}{rcl} f(m,j,1) & = & m+mj+1, \\ f(i,k,1) & = & f(i+1,k,1)+1 \text{ for } i=1,2,\ldots,m-1, \\ f(m,k,1) & = & f(1,k+1,1)+1 \text{ for } k=1,2,\ldots,j-1, \\ f(1,1,1) & = & m+2mj. \end{array}$$

(ii) (m even, j even):

Consider the following sequence:

$$\begin{array}{cccc} f(m,j-1,1), & f(m-1,j-1,1), & \cdots & f(1,j-1,1), \\ f(m,j-1,2), & f(m-1,j-1,2), & \cdots & f(1,j-1,2), \\ f(m,j-3,1), & f(m-1,j-3,1), & \cdots & f(1,j-3,1), \\ f(m,j-3,2), & f(m-1,j-3,2), & \cdots & f(1,j-3,2), \\ \vdots & \vdots & \cdots & \vdots \\ f(m,1,1), & f(m-1,1,1), & \cdots & f(1,1,1), \\ f(m,1,2), & f(m-1,1,2), & \cdots & f(1,1,2). \end{array}$$

We claim that the mj numbers in the sequence are $m+1, m+2, \ldots, m+mj$, which can be observed from the following:

(a) The first number,

$$\begin{array}{rcl} f(m,j-1,1) & = & h(x_m) - h(x_{m+(m-1)j+(j-1)}) = (2mj+2m) \\ & & -(m+mj+(2m-1)+(j/2-1)2m) = m+1. \end{array}$$

(b) For $i=1,2,\ldots,m-1$ and $1 \leqslant t \leqslant j/2$,

$$\begin{array}{rcl} f(i,2t-1,1) & = & h(x_i) - h(x_{m+(i-1)j+(2t-1)}) \\ & = & (2mj+m+i) - (m+mj+(2i-1)+(t-1)2m) \\ & = & (2mj+m+i+1) - 1 \\ & & -(m+mj+(2(i+1)-1)+(t-1)2m)+2 \\ & = & f(i+1,2t-1,1)+1. \end{array}$$

Similarly, for $i = 1, 2, \ldots, m-1$ and $1 \leqslant t \leqslant j/2$,

$$f(i, 2t-1, 2) = f(i+1, 2t-1, 2) + 1.$$

(c) For $t = 1, 2, \ldots, j/2$,

$$\begin{aligned} f(m, 2t-1, 2) &= h(y_m) - h(y_{m+(m-1)j+(2t-1)}) \\ &= (2mj + 2m) - (mj + (2m-1) + (t-1)2m) \\ &= (2mj + m + 1) \\ &\quad -(m + mj + (2-1) + (t-1)2m) + 1 \\ &= h(x_1) - h(x_{m+(1-1)j+(2t-1)}) + 1 \\ &= f(1, 2t-1, 1) + 1. \end{aligned}$$

(d) For $t = 1, 2, \ldots, (j-2)/2$,

$$\begin{aligned} f(m, 2t-1, 1) &= h(x_m) - h(x_{m+(m-1)j+(2t-1)}) \\ &= (2mj + 2m) - (m + mj + (2m-1) + (t-1)2m) \\ &= (2mj + m + 1) + m - 1 - m \\ &\quad -(mj + (2-1) + ((t+1)-1)2m) + 2 \\ &= h(y_1) - h(y_{m+(1-1)j+(2t+1))}) + 1 \\ &= f(1, 2t+1, 2) + 1. \end{aligned}$$

(e) The last number,

$$\begin{aligned} f(1, 1, 2) &= x_1 - x_{m+(1-1)j+1} \\ &= (2mj + m + 1) - (mj + (2-1) + (1-1)2m) = m + mj. \end{aligned}$$

Similarly, the mj numbers in the sequence

$$\begin{array}{cccc} f(m, j, 1), & f(m-1, j, 1), & \cdots & f(1, j, 1), \\ f(m, j, 2), & f(m-1, j, 2), & \cdots & f(1, j, 2), \\ f(m, j-2, 1), & f(m-1, j-2, 1), & \cdots & f(1, j-2, 1), \\ f(m, j-2, 2), & f(m-1, j-2, 2), & \cdots & f(1, j-2, 2), \\ \vdots & \vdots & \cdots & \vdots \\ f(m, 2, 1), & f(m-1, 2, 1), & \cdots & f(1, 2, 1), \\ f(m, 2, 2), & f(m-1, 2, 2), & \cdots & f(1, 2, 2). \end{array}$$

represent the numbers

$$m + mj + 1, m + mj + 2, \ldots, m + 2mj,$$

since

$$
\begin{aligned}
f(m,j,1) &= m+mj+1,\\
f(i,2t,1) &= f(i+1,2t,1)+1 \text{ for } i=1,2,\dots,m-1,\\
f(i,2t,2) &= f(i+1,2t,2)+1 \text{ for } i=1,2,\dots,m-1,\\
f(m,2t,2) &= f(1,2t,1)+1 \text{ for } t=1,2,\dots,j/2,\\
f(m,2t,1) &= f(1,2t+2,2)+1 \text{ for } t=1,2,\dots,(j-2)/2,\\
f(1,2,2) &= m+2mj.
\end{aligned}
$$

(*iii*) (m even, j odd):

It can be shown as in the previous case that the $m(j+1)$ numbers in the sequence

$$
\begin{array}{cccc}
f(m,j,1), & f(m-1,j,1), & \cdots & f(1,j,1),\\
f(m,j,2), & f(m-1,j,2), & \cdots & f(1,j,2),\\
f(m,j-2,1), & f(m-1,j-2,1), & \cdots & f(1,j-2,1),\\
f(m,j-2,2), & f(m-1,j-2,2), & \cdots & f(1,j-2,2),\\
\vdots & \vdots & \cdots & \vdots\\
f(m,1,1), & f(m-1,1,1), & \cdots & f(1,1,1),\\
f(m,1,2), & f(m-1,1,2), & \cdots & f(1,1,2).
\end{array}
$$

represent the numbers

$$m+1, m+2, \dots, 2m+mj,$$

since

$$
\begin{aligned}
f(m,j,1) &= m+1,\\
f(i,2t-1,1) &= f(i+1,2t-1,1)+1\\
&\quad \text{for } i=1,2,\dots,m-1 \text{ and } 1\leqslant t\leqslant (j+1)/2,\\
f(i,2t-1,2) &= f(i+1,2t-1,2)+1\\
&\quad \text{for } i=1,2,\dots,m-1 \text{ and } 1\leqslant t\leqslant (j+1)/2,\\
f(m,2t-1,2) &= f(1,2t+1,1)+1 \text{ for } t=1,2,\dots,(j-1)/2,\\
f(m,2t-1,1) &= f(1,2t+1,2)+1 \text{ for } t=1,2,\dots,(j-3)/2,\\
f(1,2,2) &= 2m+2mj.
\end{aligned}
$$

And, the $m(j-1)$ numbers in the sequence

$$\begin{array}{cccc} f(m,j-1,1), & f(m-1,j-1,1), & \cdots & f(1,j-1,1), \\ f(m,j-1,2), & f(m-1,j-1,2), & \cdots & f(1,j-1,2), \\ f(m,j-3,1), & f(m-1,j-3,1), & \cdots & f(1,j-3,1), \\ f(m,j-3,2), & f(m-1,j-3,2), & \cdots & f(1,j-3,2), \\ \vdots & \vdots & \cdots & \vdots \\ f(m,2,1), & f(m-1,2,1), & \cdots & f(1,2,1), \\ f(m,2,2), & f(m-1,2,2), & \cdots & f(1,2,2). \end{array}$$

represent the numbers

$$2m+mj+1, 2m+mj+2, \ldots, m+2mj,$$

since

$$\begin{array}{rcl} f(m,j-1,1) & = & 2m+mj+1, \\ f(i,2t,1) & = & f(i+1,2t,1)+1 \\ & & \text{for } i=1,2,\ldots,m-1 \text{ and } 1 \leqslant t \leqslant (j-1)/2, \\ f(i,2t,2) & = & f(i+1,2t,2)+1 \\ & & \text{for } i=1,2,\ldots,m-1 \text{ and } 1 \leqslant t \leqslant (j-1)/2, \\ f(m,2t,2) & = & f(1,2t,1)+1 \text{ for } t=1,2,\ldots,(j-1)/2, \\ f(m,2t,1) & = & f(1,2t+2,2)+1 \text{ for } t=1,2,\ldots,(j-3)/2, \\ f(1,2,2) & = & m+2mj. \end{array}$$

□

Theorem 2. For $m \geqslant 3$, $(\mathcal{C}'_{m,j})_\alpha = 2$ where $j \geqslant 1$.

Proof. The proof follows from Lemmas 1 and 2, and Observation 1. □

3 Trees with α-deficits

In this section, we have relied on the results of Brinkmann et al. in [1].

Conjecture 2. If $\Delta_T = 2k+1$, then $\alpha_{def}(T) \leqslant k$.

Conjecture 3. For all $k \geqslant 1$ and for all $2 \leqslant j \leqslant 2k$,

$$\alpha_{def}(\mathcal{C}'_{2k+1,j}) = k.$$

Lemma 3. For $k \geqslant 1$ and $2 \leqslant j \leqslant 2k$,

$$\alpha_{def}\left(\mathcal{C}'_{2k+1,j}\right) \leqslant k.$$

Proof. Consider the graph $\mathcal{C}'_{m,j}$ with $m = 2k+1$. Let the vertices be

$$x_0, x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+mj}$$

where x_0 is the central vertex with degree m, each of the vertices $x_1, x_2, \ldots, x_m$ has degree $j+1$, and $x_{m+1}, x_{m+2}, \ldots, x_{m+mj}$ are the pendant vertices. Consider the vertex labeling h with $h(x_0) = 0$, $h(x_i) = mj+i$ for $i = 1, 2, \ldots, m$ and

$$h(x_{m+(i-1)j+r}) \quad = \quad \begin{cases} (2i-1)+(r-1)m, & \text{for } 1 \leqslant i \leqslant m,\ 1 \leqslant r \leqslant j-1; \\ (2i-1)+(j-1)m, & \text{for } 1 \leqslant i \leqslant m-k. \end{cases}$$

Similar to the m-odd case of Lemma 1, the vertices

$$x_0, x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+mj-k}$$

have distinct labels from $0, 1, 2, \ldots, m+mj$. Similar to the m-odd case of Lemma 2, all edges have distinct labels except that the labels for the k edges $(x_i, x_{m+(i-1)j+j})$ with $i = m-k+1, m-k+2, \ldots, m$ are missing. $\square$

Proposition 1. For $k \geqslant 1$ and $2 \leqslant j \leqslant 2k$,

$$\alpha_{def}\left(\mathcal{C}'_{2k+1,j}\right) > 0.$$

Proof. Let $G = \mathcal{C}'_{m,j}$ where $m = 2k+1$ with vertices

$$x_0, x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+mj}$$

where x_0 is the central vertex with degree m and $x_{m+1}, x_{m+2}, \ldots, x_{m+mj}$ are the pendant vertices. Assume that G has an α-labeling ℓ. Then, the sum of all edge-labels,

$$S = \sum_{i=1}^{m+mj} i = (m+mj)(m+mj+1)/2 \equiv 0 \pmod{m}.$$

By Remark $B1$ of Brinkmann et al. [1], let the vertices x_i for $i = 1, 2, \ldots, m$ be labeled with $mj+i$, respectively. The remaining numbers $0, 1, 2, \ldots, mj$ label x_0 and the pendant vertices. For any choice of $\ell(x_0) \in \{0, 1, 2, \ldots, mj\}$, we have

$$\begin{aligned} S_1 \quad &= \quad \sum_{i=1}^{m} (\ell(x_i) - \ell(x_0)) = \sum_{i=1}^{m} (\ell(x_i)) - \sum_{i=1}^{m} (\ell(x_0)) \\ &= \quad m^2 j + m(m+1)/2 - m\ell(x_0) \\ &\equiv \quad 0 \pmod{m}. \end{aligned}$$

Since ℓ is an α-labeling, for $i = 1, 2, \ldots, m$ and $t = 1, 2, \ldots, j$, the pendant vertices $x_{m+(i-1)j+t}$ are labeled in such a way that

$$\begin{aligned} S_2 &= \sum_{i=1}^{m}\sum_{t=1}^{j}\big(\ell(x_i) - \ell(x_{m+(i-1)j+t})\big) \\ &= j\sum_{i=1}^{m}\ell(x_i) - \sum_{i=1}^{m}\sum_{t=1}^{j}\big(\ell(x_{m+(i-1)j+t})\big) \\ &\equiv 0 \pmod m, \text{ (since } S = S_1 + S_2 \text{ and } S, S_1 \equiv 0 \pmod m) \end{aligned}$$

implying

$$\sum_{i=1}^{m}\sum_{t=1}^{j}\big(\ell(x_{m+(i-1)j+t})\big) \equiv 0 \pmod m,$$

which holds only if $\ell(x_0)$ is chosen from the $mj+1$ labels $0, 1, \ldots, mj$ in such a way that

$$\ell(x_0) \equiv 0 \pmod m.$$

Suppose $\ell(x_0) = 0$. Then the edge-labels of (x_0, x_i) for $i = 1, 2, \ldots, m$ are

$$mj + 1, mj + 2, \ldots, mj + m.$$

Since the labels less than $mj + 1$ must still be used, we may determine the following locations for vertex labels in order:

1 can only label a vertex attached to x_1, adding mj to the set of edge-labels,

2 can only label a vertex attached to x_1, adding $mj - 1$ to the set of edge-labels,

$\vdots$

j can only label a vertex attached to x_1, adding $mj - (j - 1)$ to the set of edge-labels.

All the pendant vertices attached to x_1 are labeled and $j + 1$ cannot be used to label any pendant vertex attached to x_i for $i = 2, 3, \ldots, m$. Hence $\ell(x_0) \neq 0$.

Let $\ell(x_0) = tm$ with $1 \leqslant t \leqslant j$. Then the edge-labels of (x_0, x_i) for $i = 1, 2, \ldots, m$ are

$$mj - mt + 1, mj - mt + 2, \ldots, mj - mt + m.$$

As above, we may determine the locations of certain vertex labels:

The only way to add edge-label $m + mj$ is to use 0 to label a vertex attached to x_m,

The only way to add edge-label $m + mj - 1$ is to use 1 to label a vertex attached to x_m,

$\vdots$

The only way to add edge-label $m + mj - (j-1)$ is to use $j-1$ to label a vertex attached to x_m.

All the pendant vertices attached to x_m are labeled and the labels used are $0, 1, 2, \ldots, j-1$. But the only way to add the edge-label $m + mj - j$ is to use $r \in \{0, 1, 2, \ldots, j-1\}$ to label a pendant vertex attached to x_{m-j+r} so that

$$\ell(x_{m-j+r}) - r = mj + (m - j + r) - r = m + mj - j,$$

which is impossible. Hence, we have a contradiction to our assumption that G has an α-labeling. □

4 Concluding remarks

In this paper, we have given an example of constructing a graph with a graceful, bipartite labeling which can be decomposed into two isomorphic edge-disjoint trees consisting of a root node of degree m, each of whose neighbours is connected to j ($j \geqslant 1$) leaves.

This result is a special case of the conjecture that for every tree T, two copies of T can be packed into a graph with a graceful, bipartite labeling. The result remotely connects to the graceful tree conjecture which states that all trees are graceful. We have also explored the extent to which a bipartite labeling falls short of gracefulness.

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