On Generalized Schur Numbers

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Abstract

Let $\mathcal{L}(t)$ represent the equation $x_1 + x_2 + \cdots + x_{t-1} = x_t$. For $k \geqslant 1$, $0 \leqslant i \leqslant k-1$, and $t_i \geqslant 3$, the *generalized Schur number* $S(k; t_0, t_1, \ldots, t_{k-1})$ is the least positive integer m such that for every k-colouring of $\{1, 2, \ldots, m\}$, there exists an $i \in \{0, 1, \ldots, k-1\}$ such that there exists a solution to $\mathcal{L}(t_i)$ that is monochromatic in colour i. In this paper, we report twenty-six previously unknown values of $S(k; t_0, t_1, \ldots, t_{k-1})$ and conjecture that for $4 \leqslant t_0 \leqslant t_1 \leqslant t_2$, $S(3; t_0, t_1, t_2) = t_2 t_1 t_0 - t_2 t_1 - t_2 - 1$.

1 Introduction

For integers a and b, let $[a, b]$ denote the interval $\{x : a \leqslant x \leqslant b\}$. The Schur number $S(k)$ is the smallest positive integer n such that for every 2-colouring of $[1, n]$, there is a monochromatic solution to $x + y = z$ with $y \geqslant x$. Schur [13] proved that $S(k)$ is finite. Let $\mathcal{L}(t)$ represent the equation $x_1 + x_2 + \cdots + x_{t-1} = x_t$. For $k \geqslant 1$, $0 \leqslant i \leqslant k-1$, and $t_i \geqslant 3$, the *generalized Schur number* $S(k; t_0, t_1, \ldots, t_{k-1})$ is the least positive integer m such that for every k-colouring of $[1, m]$, there exists an $i \in \{0, 1, \ldots, k-1\}$ such that there exists a solution to $\mathcal{L}(t_i)$ that is monochromatic in colour i. The *generalized diagonal Schur number* $S(k, t)$ equals $S(k; t_0, t_1, \ldots, t_{k-1})$ where $t_0 = t_1 = \cdots = t_{k-1} = t$. If the values of t_i are not all equal, then $S(k; t_0, t_1, \ldots, t_{k-1})$ is called the *generalized off-diagonal Schur number*. This name refers to the similarity with the off-diagonal Ramsey numbers. Also, $S(k)$ equals $S(k, 3) = S(k; 3, 3, \ldots, 3)$. For different variations, results, and references, see Schaal [11], Bialostocki and Schaal [4], Fredricksen and Sweet [6], Landman and Robertson [8], Martinelli and Schaal [9], Schaal and Snevily [12], Kézdy et. al [7], and Ahmed et al. [2].

It is known that $S(2, 3) = 5$, $S(3, 3) = 14$, and $S(4, 3) = 45$. Beutelspacher and Brestovansky [3] determined $S(2; t, t) = t^2 - t - 1$ and Robertson and Schaal [10]) proved for $s, t \geqslant 3$,

$$S(2; s, t) = \begin{cases} 3t - 4, & \text{if } s = 3 \text{ and } t \equiv 1 \pmod 2; \\ 3t - 5, & \text{if } s = 3 \text{ and } t \equiv 0 \pmod 2; \\ st - t - 1 & \text{if } 4 \leqslant s \leqslant t. \end{cases}$$

The known lower bound for generalized diagonal Schur numbers is

$$S(k, t) \geqslant \frac{t^{k+1} - 2t^k + 1}{t - 1} = t^k - t^{k-1} - t^{k-2} - \cdots - t - 1.$$

An interesting open problem is to compute the exact value of $S(5,3)$. The current lower bound $S(5,3) \geqslant 161$ (Exoo [5]) is almost twenty years old. Many numbers in the general form are yet unknown. In this paper, we report twenty-six previously unknown values of $S(k;t_0,t_1,\ldots,t_{k-1})$ including three generalized diagonal Schur numbers. Based on experiemental data, we conjecture that for $4 \leqslant t_0 \leqslant t_1 \leqslant t_2$, $S(3;t_0,t_1,t_2) = t_2t_1t_0 - t_2t_1 - t_2 - 1$.

2 Some new generalized Schur numbers

Let $f : \{1,2,\ldots,n\} \to \{0,1,\ldots,k-1\}$ denote a colouring of the numbers $1,2,\ldots,n$ with the k colours $0,1,\ldots,k-1$. A colouring is *valid* with respect to the Schur number $S(k;t_0,t_1,\ldots,t_{k-1})$ if there exists no solution to $\mathcal{L}(t_i)$ monochromatic in colour i for any $i \in \{0,1,\ldots,k-1\}$ under f. The existence of a valid colouring of $[1,n]$ proves that $S(k;t_0,t_1,\ldots,t_{k-1}) > n$.

2.1 Computer assisted results

In this section, we report some new generalized Schur numbers with enumeration of all valid colourings for each number. The valid colourings are determined using exhaustive computer search algorithms.

Theorem 2.1. $S(3;3,3,4) = 23$.

Proof. The eighteen valid colourings of $[1,22]$ are

(a) $01102021a2222b12020110$ where $(a,b) \in \{0,1,2\}^2$, and

(b) The nine colourings obtained from (a) using the permutation $(0,1)(2)$.

In (a):
If $f(23) = 0$, then there is a solution $1 + 22 = 23$ with colour 0.
If $f(23) = 1$, then there is a solution $2 + 21 = 23$ with colour 1.
If $f(23) = 2$, then there is a solution $5 + 5 + 13 = 23$ with colour 2.
Similarly, the colourings in (b) cannot be extended. Hence, $S(3;3,3,4) \leqslant 23$. □

Theorem 2.2. $S(3;3,3,5) = 32$.

Proof. The fifty-four valid colourings of $[1,31]$ are

(a) $01^2 0201202a1222b2221c20210201^2 0$ where $(a,b,c) \in \{0,1,2\}^3$, and

(b) The twenty-seven colourings obtained from (a) using the permutation $(0,1)(2)$.

In (a):
If $f(32) = 0$, then there is a solution $1 + 31 = 32$ with colour 0.
If $f(32) = 1$, then there is a solution $2 + 30 = 32$ with colour 1.
If $f(32) = 2$, then there is a solution $5 + 5 + 5 + 17 = 32$ with colour 2.
Similarly, the colourings in (b) cannot be extended. Hence, $S(3;3,3,5) \leqslant 32$. □

Theorem 2.3. $S(3;3,3,6) = 41$.

Proof. The one hundred and sixty-two valid colourings of $[1,40]$ are

(a) $011020112201a2202b2222c2022d102211020110$ where $(a,b,c,d) \in \{0,1,2\}^4$; and

(b) the eighty-one colourings obtained from (a) using the permutation $(0,1)(2)$.

In (a):

If $f(41) = 0$, then there is a solution $1 + 40 = 41$ with colour 0.
If $f(41) = 1$, then there is a solution $2 + 39 = 41$ with colour 1.
If $f(41) = 2$, then there is a solution $5 + 5 + 5 + 5 + 21 = 41$ with colour 2.
Similarly, the colourings in (b) cannot be extended. Hence, $S(3; 3, 3, 6) \leqslant 41$. □

Theorem 2.4. $S(3; 3, 3, 7) = 49$.

Proof. The eight hundred and forty-six valid colourings of $[1, 48]$ are

(a) $0110201102211022a2022b2222c220de2201f22011020110$ with $(a, b, c) \in \{0, 1, 2\}^3$ and

$$(d, e, f) \in \{(1,0,1),(1,0,2),(1,1,1),(1,1,2),(1,2,1),(1,2,2),\\ (2,0,1),(2,0,2),(2,1,1),(2,1,2),(2,2,1),(2,2,2)\};$$

(b) $01020102a102120212^31221122b2^3c202d201e201f2010$ with $(a, b, c, d, e, f) \in \{0, 1\}^6$;

(c) $01020102a102020212^3122112212^3b202c201d201e2010$ with $(a, b, c, d, e) \in \{0, 1\}^5$;

(d) $0102010211021202122^3122102212^3120a12011201b2010$ with $(a, b) \in \{(1, 0), (2, 0), (2, 1)\}$; and

(e) the four hundred and twenty-three colourings obtained from $(a) - (d)$ using the permutation $(0, 1)(2)$.

In $(a) - (d)$:

If $f(49) = 0$, then there is a solution $1 + 48 = 49$ with colour 0.
If $f(49) = 1$, then there is a solution $2 + 47 = 49$ with colour 1.

In $(b) - (d)$:

If $f(49) = 2$, then there is a solution $4 + 4 + 4 + 4 + 4 + 29 = 49$ with colour 2;

In (a):

If $f(49) = 2$, then there is a solution $5 + 5 + 5 + 5 + 5 + 24 = 49$ with colour 2;
Similarly, the colourings in (e) cannot be extended. Hence, $S(3; 3, 3, 7) \leqslant 49$. □

Theorem 2.5. $S(3; 3, 4, 4) = 31$.

Proof. The eight valid colourings of $[1, 30]$, any one of which proves that $S(3; 3, 4, 4) \geqslant 31$ are

(a) $0101020202121222212a2020201010$ with $a \in \{0, 1\}$;

(b) $0111102020222222222220202011a10$ with $a \in \{0, 1\}$; and

(c) the 4 colourings obtained from (a) and (b) using the permutation $(0)(1, 2)$.

In (a) and (b):

If $f(31) = 0$, then there is a solution $1 + 30 = 31$ with colour 0.
If $f(31) = 1$, then there is a solution $2 + 2 + 27 = 31$ with colour 1.
If $f(31) = 2$, then there is a solution $6 + 6 + 19 = 31$ with colour 2.
Similarly, the four (4) colourings in (c) cannot be extended. Hence, $S(3; 3, 4, 4) \leqslant 31$. □

Theorem 2.6. $S(3; 3, 4, 5) = 47$.

Proof. The only valid colouring of $[1, 46]$ is

$$0202^402010101^{18}01010202^4020.$$

If $f(47) = 0$, then there is a solution $1 + 46 = 47$ with colour 0.
If $f(47) = 1$, then there is a solution $11 + 13 + 23 = 47$ with colour 1.
If $f(47) = 2$, then there is a solution $2 + 2 + 2 + 41 = 47$ with colour 2.
Hence, $S(3; 3, 4, 5) \leqslant 47$. □

Theorem 2.7. $S(3;3,5,5) = 58$.

Proof. The one hundred and twelve valid colourings of $[1,57]$ are

(a) $0101^4 01020202^{12} 02a2b2c2^{10} 02020101^4 010$ with $(a,b,c) \in \{0,2\}^3$;

(b) $0101^4 01020202^{14} 02a2b2c2^{8} 02020101^4 010$ with $(a,b,c) \in \{0,2\}^3$;

(c) $0101^4 01020202^{16} 02a2b2c2^{6} 02020101^4 010$ with $(a,b,c) \in \{0,2\}^3$;

(d) $0101^4 01020202^{18} a2b2c2d2^{4} 02020101^4 010$ with $(a,b,c,d) \in \{0,2\}^4$;

(e) $0101^4 01020202^{4} 02a2^1 3b2c2d2^{4} 02020101^4 010$ with

$$(a,b,c,d) \in \{(0,2,0,0),(0,2,0,2),(0,2,2,0),(0,2,2,2),(2,0,0,0),(2,0,0,2),(2,0,2,0),(2,0,2,2)\};$$

(f) $0101^4 01020202^{4} a2b2^{13} 22c2d2^{4} 02020101^4 010$ with

$$(a,b,c,d) \in \{(0,2,0,0),(0,2,0,2),(0,2,2,0),(0,2,2,2),(2,0,0,0),(2,0,0,2),(2,0,2,0),(2,0,2,2)\};$$

(g) the fifty-six colourings obtained from $(a)-(f)$ using the permutation $(0)(1,2)$.

In each of $(a),(b),\ldots,(f)$:
If $f(58) = 0$, then there is a solution $1 + 57 = 58$ with colour 0.
If $f(58) = 1$, then there is a solution $2 + 2 + 2 + 52 = 58$ with colour 1.
If $f(58) = 2$, then there is a solution $11 + 11 + 11 + 25 = 58$ with colour 2.
Similarly, the colourings in (g) also cannot be extended. Hence, $S(3;3,5,5) \leqslant 58$. □

Theorem 2.8. $S(3;4,4,4) = 43$.

Proof. The ninety-six valid colourings of $[1,42]$ are

(a) $0^2 1^6 0^2 2^6 ab2^6 cd2^6 0^2 1^6 0^2$ where $(a,b,c,d) \in \{0,2\}^4$; and

(b) the eighty colourings obtained from (a) using the permutations $(0)(1,2)$, $(0,1)(2)$, $(0,1,2)$, $(0,2,1)$, and $(0,2)(1)$.

In (a):
If $f(43) = 0$, then we have a solution $1 + 1 + 41 = 43$ with colour 0.
If $f(43) = 1$, then we have a solution $3 + 3 + 35 = 43$ with colour 1.
If $f(43) = 2$, then we have a solution $11 + 11 + 21 = 43$ with colour 2.
So, the colouring cannot be extended. Similarly, the other cases also cannot be extended. Hence, $S(3;4,4,4) \leqslant 43$. □

Theorem 2.9. $S(3;4,4,5) = 54$.

Proof. The three thousand five hundred and eighty-four valid colourings of $[1,53]$ are

(a) $0^2 1^6 0^2 2^5 a2^6 b2cdefg2h2^5 ijk2^4 0^2 1^6 0^2$ with $a,b,h \in \{0,2\}$, $(c,d,e,f,g) \in \{1,2\}^5$, and $(i,j,k) \in \{(0,0,2),(0,2,2),(2,0,0),(2,0,2),(2,2,0),(2,2,2)\}$ (1536 solutions);

(b) $0^2 1^6 0^2 2^{12} a2bcdef2g2^5 hi2^5 0^2 1^6 0^2$ with $a,g \in \{0,2\}$, $(b,c,d,e,f) \in \{1,2\}^5$, and $(h,i) \in \{(0,0),(0,2)\}$ (256 solutions); and

(c) the one thousand seven hundred and ninety-two colourings obtained from (a) and (b) with the permutation $(0,1)(2)$.

In both (a) and (b):

If $f(54) = 0$, then there is a solution $1 + 1 + 52 = 54$ with colour 0.

If $f(54) = 1$, then there is a solution $3 + 3 + 48 = 54$ with colour 1.

If $f(54) = 2$, then there is a solution $11 + 11 + 11 + 21 = 54$ with colour 2.

Similarly, the other one thousand seven hundred and ninety-two solutions cannot be extended. Hence, $S(3; 4, 4, 5) \leqslant 54$. $\square$

Theorem 2.10. $S(3; 4, 5, 5) = 69$.

Proof. The nine thousand four hundred and eighty-eight valid colourings of $[1, 68]$ are

(a) $0^2 1^4 x 1^4 0^2 2^4 a 2^4 bc 2^4 d 2^4 ef 2^4 g 2^4 hi 2^4 j 2^4 0^2 1^4 y 1^4 0^2$ with $(a, b, c, d, e, f, g, h, i, j) \in \{0, 2\}^{10}$ and $(x, y) \in \{0, 1\}^2$ (4096 colourings);

(b) $0^2 1^4 x 1^4 0^2 2^4 a 2^4 b 2^5 c 2^4 d 2^5 e 2^4 f 2^4 0 g 2^4 0^2 1^4 y 1^4 0^2$ with $(a, b, c, d, e, f, g) \in \{0, 1\}^7$ and $(x, y) \in \{0, 1\}^2$ (512 colourings);

(c) $0^2 1^9 0^2 2^{10} 0 2^4 a 2^5 b 2^4 c 2^5 d 0 2^8 0^2 1^9 0^2$ with $(a, b, c, d) \in \{0, 2\}^4$ (16 colourings);

(d) $0^2 1^9 0^2 2^6 ab 2^7 c 2^{10} d 2^7 ef 2^6 0^2 1^9 0^2$ with $(c, d, e, f) \in \{0, 2\}^4$ and $(a, b) \in \{00, 02, 20\}$ (48 colourings);

(e) $0^2 1^9 0^2 2^{15} a 2^{10} b 2^7 cd 2^6 0^2 1^9 0^2$ with $(a, b) \in \{0, 2\}^2$ and $(c, d) \in \{00, 02, 20\}$ (12 colourings);

(f) $0^2 1^9 0^2 2^{15} a 2^5 b 2^4 c 2^5 d 0 2^8 0^2 1^9 0^2$ with $(a, b, c, d) \in \{0, 2\}^4$ (16 colourings);

(g) $0^2 1^9 0^2 2^6 0^2 2^{18} a 2^8 bcd 2^4 0^2 1^9 0^2$ with $a \in \{0, 2\}$ and $(b, c, d) \in \{(0, 0, 2), (2, 0, 0), (2, 0, 2), (2, 2, 0)\}$ (8 colourings);

(h) $0^2 1^9 0^2 2^6 ab 2^{27} cde 2^4 0^2 1^4 f 1^4 0^2$ with $(a, b) \in \{(0, 0), (2, 0)\}$ and

$$(c, d, e, f) \in \{(0, 0, 2, 1), (2, 0, 0, 0), (2, 0, 0, 1), (2, 0, 2, 0), (2, 0, 2, 1), (2, 2, 0, 0), (2, 2, 0, 1), (2, 2, 2, 0)\}$$

(16 colourings);

(i) $0^2 1^9 0^2 2^{15} a 2^{10} b 2^6 cdef 2^5 0^2 1^9 0^2$ with $(a, b) \in \{0, 2\}^2$ and $(c, d, e, f) \in \{(0, 0, 2, 2), (2, 2, 0, 0)\}$ (8 colourings);

(j) $0^2 1^9 0^2 2^7 0 2^7 a 2^{10} b 2^8 cde 2^4 0^2 1^9 0^2$ with

$$(a, b, c, d, e) \in \{(0, 0, 0, 0, 2), (0, 0, 2, 0, 2), (0, 2, 0, 0, 2), (0, 2, 2, 0, 2), (2, 0, 0, 0, 2), (2, 0, 2, 0, 0), (2, 0, 2, 0, 2), (2, 0, 2, 2, 0)\}$$

(8 colourings);

(k) $0^2 1^9 0^2 2^6 0 2^{19} 0 2^8 abc 2^4 0^2 1^9 0^2$ with $(a, b, c) \in \{(0, 0, 2), (2, 0, 0), (2, 0, 2), (2, 2, 0)\}$ (4 colourings);

(l) the four thousand seven hundred and forty-four colourings obtained from (a) $-$ (k) with the permutation $(0)(1, 2)$.

In each of $(a), (b), \ldots, (k)$:

If $f(69) = 0$, then there is a solution $1 + 1 + 67 = 69$ with colour 0.

If $f(69) = 1$, then there is a solution $3 + 3 + 3 + 60 = 69$ with colour 1.

If $f(69) = 2$, then there is a solution $14 + 14 + 14 + 27 = 69$ with colour 2.

Similarly, the other four thousand seven hundred and forty-four solutions cannot be extended. Hence, $S(3; 4, 5, 5) \leqslant 69$. $\square$

2.2 Computer generated results

In this section, for each generalized Schur number $n = S(k, t_0, t_1, \ldots, t_{k-1})$, we provide a single valid colouring of $[1, n-1]$ as a proof of the lower bound $S(k, t_0, t_1, \ldots, t_{k-1}) \geqslant n$. To determine exactness, we use complete SAT solvers, where we construct an SAT instance of the Schur problem in such a way that the SAT instance corresponding to $k, t_0, t_1, \ldots, t_{k-1}$ and m is satisfiable if and only if $S(k, t_0, t_1, \ldots, t_{k-1}) > m$. For an introduction to the SAT problem and ideas for encoding similar problems to SAT, see Ahmed [1].

- $S(3; 3, 4, 6) = 49$: $0(10)^2(20)^5 1^4 2^{10} 1(10)^2(20)^5(10)^2$.
- $S(3; 3, 4, 7) = 59$: $0(10)^2(20)^4 10(20)^2(21)^2(12)^2 2^5 121^2 21(20)^3 10(20)^4(10)^2$.
- $S(3; 3, 5, 6) = 70$: $0202^6 020(10)^3 1^{18} 01^5 01^8 0(10)^3 202^6 020$.
- $S(3; 3, 5, 7) = 80$: $0101^4 010(20)^2 2^4 0(20)^2 2^8 02^5 121^3 2^5 02^{17} 0(20)^2 101^4 010$.
- $S(3; 3, 6, 6) = 85$: $0(10)^4(20)^9(21)^4 2^{14}(12)^3 0(20)^{10}(10)^4$.
- $S(3; 3, 6, 7) = 107$: $0(20)^2 2^6 0(20)^2(10)^3 1^{18} 01^7 01^6 0(10)^2 1^3 01^{20} 0(10)^3(20)^2 2^6 0(20)^2$.
- $S(3; 4, 4, 6) = 65$: $1^2 0^6 1^2 2^{13} 1^2 2^5 01202^{12} 12^7 1^2 0^6 1^2$.
- $S(3; 4, 4, 7) = 76$: $0^2 1^6 0^2 2^7 02^7 02^8 01^3 2^{19} 02^7 0^2 1^6 0^2$.
- $S(3; 4, 5, 6) = 83$: $0^2 1^9 0^2 2^6 02^{49} 0^2 1^9 0^2$.
- $S(3; 4, 5, 7) = 97$: $0^2 1^9 0^2 2^4 02^5 02^9 02^5 02^4 02^2 12^8 02^4 02^4 0^2 2^5 02^4 02^4 0^2 1^9 0^2$.
- $S(3; 4, 6, 6) = 101$: $0^2 1^{12} 0^2 2^8 0^2 2^{36} 02^{21} 0^2 1^{12} 0^2$.
- $S(3; 5, 5, 5) = 94$: $0^3 1^{12} 0^3 2^{12} 0^3 2^{12} 0^3 2^{12} 0^3 2^{12} 0^3 1^{12} 0^3$.
- $S(3; 5, 5, 6) = 113$: $0^3 1^{12} 0^3 2^{64} 0^2 2^{10} 0^3 1^{12} 0^3$.
- $S(3; 6, 6, 6) = 173$: $0^4 1^{20} 0^4 2^{20} 0^4 2^{20} 0^4 2^{20} 0^4 2^{20} 0^4 2^{20} 0^4 1^{20} 0^4$.
- $S(4; 3, 3, 3, 4) = 77$: $01^2 02^5 01^2 0301^2 02^2 03203^2 03^5 231^2 3^4 1^2 323^5 03^2 02^2 12^2 01^2 0301^2 02^2 02^2 01^2 0$.
- $S(4; 3, 3, 3, 5) = 107$:

$$120^2 12^2 10^2 213120^2 12310^2 2131020123103^5 023^4 103^3 23^2 23^3 013^2$$

$$13201313^2 0132102013120^2 13210^2 213120^2 12^2 10^2 21.$$

2.3 A lower bound for the generalized off-diagonal Schur numbers

Theorem 2.11. For all $k \geqslant 1$ and for all $t_0, t_1, \ldots, t_{k-1}$ where $3 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_{k-1}$,

$$S(k; t_0, t_1, \ldots, t_{k-1}) \quad \geqslant \quad \prod_{j=0}^{k-1} t_j - \sum_{i=1}^{k-1} \prod_{j=i}^{k-1} t_j - 1.$$

Proof. We will use induction on k. When $k = 1$, it is clear that $S(1; t_0) = t_0 - 1$. Let $k_0 \geqslant 1$ be given and assume that the theorem is true for k_0. We will show that the theorem is true for $k_0 + 1$. That is, we let $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_{k_0}$ be given and assume that

$$S(k_0; t_0, t_1, \ldots, t_{k_0-1}) \quad \geqslant \quad \prod_{j=0}^{k_0-1} t_j - \sum_{i=1}^{k_0-1} \prod_{j=i}^{k_0-1} t_j - 1.$$

and we will show that

$$\begin{aligned}
S(k_0+1;t_0,t_1,\ldots,t_{k_0}) &\geqslant t_{k_0}\left(S(k_0:t_0,t_1,\ldots,t_{k_0-1})\right)-1\\
&\geqslant t_{k_0}\left(\prod_{j=0}^{k_0-1}t_j-\sum_{i=1}^{k_0-1}\prod_{j=i}^{k_0-1}t_j-1\right)-1\\
&= \prod_{j=0}^{k_0}t_j-\sum_{i=1}^{k_0}\prod_{j=i}^{k_0}t_j-1
\end{aligned}$$

For ease of notation, let $S(k_0;t_0,t_1,\ldots,t_{k_0-1})=\Theta$. We must exhibit a colouring

$$\Delta:\{1,2,\ldots,t_{k_0}\cdot\Theta-2\}\to\{0,1,\ldots,k_0\}$$

that avoids solutions to $L(t_i)$ that are monochromatic in colour i for every $i\in\{0,1,\ldots,k_0\}$. From the definition of $S(k_0;t_0,t_1,\ldots,t_{k_0-1})$, there exists a colouring $C:\{1,2,\ldots,\Theta-1\}\to\{0,1,\ldots,k_0-1\}$ that avoids solutions to $L(t_i)$ that are monochromatic in colour i for every $i\in\{0,1,\ldots,k_0-1\}$. Let Δ be defined by

$$\Delta(x)=\begin{cases} C(x), & \text{if} \quad 1\leqslant x\leqslant\Theta-1;\\ k_0, & \text{if} \quad \Theta\leqslant x\leqslant(t_{k_0}-1)\,\Theta-1;\\ C(x-(t_{k_0}-1)\,\Theta+1), & \text{if} \quad (t_{k_0}-1)\,\Theta\leqslant x\leqslant t_{k_0}\Theta-2.\end{cases}$$

Clearly, Δ avoids solutions to $L(t_{k_0})$ that are monochromatic in colour k_0. To show that Δ avoids solutions to $L(t_i)$ that are monochromatic in colour i where $i\in\{0,1,\ldots,k_0-1\}$, we will consider three cases. Let $i\in\{0,1,\ldots,k_0-1\}$ be given and let $(x_1,x_2,\ldots,x_{t_i})$ be a solution to $L(t_i)$ where $x_j\in\{1,2,\ldots,t_{k_0}\Theta-2\}$ for all $j\in\{1,2,\ldots,t_i\}$. Clearly, no more than one integer in the set $\{x_1,x_2,\ldots,x_{t_i-1}\}$ may be greater than $(t_{k_0}-1)\,\Theta-1$.

Case 1: Assume that $x_j\in\{1,2,\ldots,\Theta-1\}$ for every $j\in\{1,2,\ldots,t_i\}$. From the definition of the colouring C, it follows that $(x_1,x_2,\ldots,x_{t_i})$ is not monochromatic in colour i.

Case 2: Assume that $x_j\in\{1,2,\ldots,\Theta-1\}$ for every $j\in\{1,2,\ldots,t_i-1\}$ and $x_{t_i}\geqslant\Theta$. We have that $x_{t_i}=x_1+x_2+\cdots+x_{t_i-1}\leqslant(t_i-1)(\Theta-1)<(t_{k_0}-1)\,\Theta$ since $t_i\leqslant t_{k_0}$. So $\Delta(x_{t_i})=k_0$ and $(x_1,x_2,\ldots,x_{t_i})$ is not monochromatic in colour i.

Case 3: Assume there exists a $j\in\{1,2,\ldots,t_i-1\}$ such that $x_j\geqslant\Theta$. If $\Delta(x_j)=k_0$, then clearly $(x_1,x_2,\ldots,x_{t_i})$ is not monochromatic in colour i, so we may assume that $x_j\geqslant(t_{k_0}-1)\,\Theta$. Without loss of generality we may assume that

$$x_1,x_2,\ldots,x_{t_i-2}\in\{1,2,\ldots,\Theta-1\}$$

and

$$x_{t_i-1},x_{t_i}\in\{(t_{k_0}-1)\,\Theta,(t_{k_0}-1)\,\Theta+1,\ldots,t_{k_0}\Theta-2\}.$$

Now, let $y_1=x_{t_i-1}-(t_{k_0}-1)\,\Theta+1$ and $y_2=x_{t_i}-(t_{k_0}-1)\,\Theta+1$. Note that

$$y_1,y_2\in\{1,2,\ldots,\Theta-1\},$$

$$\Delta(y_1)=\Delta(x_{t_i-1})\text{ and }\Delta(y_2)=\Delta(x_{t_i})$$

and that $(x_1,x_2,\ldots,x_{t_i-2},y_1,y_2)$ is a solution to $L(t_i)$. From the definition of the coloring C, it follows that $(x_1,x_2,\ldots,x_{t_i-2},y_1,y_2)$ is not monochromatic in colour i, so it follows that $(x_1,x_2,\ldots,x_{t_i})$ is also not monochromatic in colour i. □

When $t_0=t_1=\cdots=t_{k-1}=t$, Theorem 2.11 simplifies to

$$S(k,t)\geqslant\frac{t^{k+1}-2t^k+1}{t\,-\,1}=t^k-t^{k-1}-t^{k-2}-\cdots-t-1.$$

Table 1 gives all known Schur numbers (and, for one case, the best known lower bound). New values are marked with *.

	Known Value	Lower Bound	Value - Lower Bound
$S(3;3,3,3) = S(3,3) = S(3)$	14	14	0
$S(3;3,3,4)$	23*	19	4
$S(3;3,3,5)$	32*	24	8
$S(3;3,3,6)$	41*	29	12
$S(3;3,3,7)$	49*	34	15
$S(3;3,4,4)$	31*	27	4
$S(3;3,4,5)$	47*	34	13
$S(3;3,4,6)$	49*	41	8
$S(3;3,4,7)$	59*	48	11
$S(3;3,5,5)$	58*	44	14
$S(3;3,5,6)$	70*	53	17
$S(3;3,5,7)$	80*	62	18
$S(3;3,6,6)$	85*	65	20
$S(3;3,6,7)$	107*	76	31
$S(3;4,4,4) = S(3,4)$	43*	43	0
$S(3;4,4,5)$	54*	54	0
$S(3;4,4,6)$	65*	65	0
$S(3;4,4,7)$	76*	76	0
$S(3;4,5,5)$	69*	69	0
$S(3;4,5,6)$	83*	83	0
$S(3;4,5,7)$	97*	97	0
$S(3;4,6,6)$	101*	101	0
$S(3;5,5,5) = S(3,5)$	94*	94	0
$S(3;5,5,6)$	113*	113	0
$S(3;6,6,6) = S(3,6)$	173*	173	0
$S(4;3,3,3,3)$	45	41	4
$S(4;3,3,3,4)$	77*	55	22
$S(4;3,3,3,5)$	107*	69	38
$S(5;3,3,3,3,3) = S(5,3) = S(5)$	$\geqslant 161$	122	$\geqslant 39$

Table 1: Known Schur Numbers and lower bounds from Theorem 2.11.

Based on the computed values and bounds, and Theorem 2.11, we propose the following conjectures:

Conjecture 2.1. For $4 \leqslant s \leqslant t \leqslant u$,

$$S(3; s, t, u) = stu - tu - u - 1.$$

Conjecture 2.2. For $3 = t < u$ and $3 < t \leqslant u$

$$S(3; 3, t, u) > 3tu - tu - u - 1.$$

It can be observed that the above conjectures are similar to the theorem for $S(2; s, t)$ given by Robertson and Schaal [10].

2.4 Valid colourings for $S(3, t)$

For $t \geqslant 4$, there are at least $(3!) \cdot 2^{(t-2)^2}$ valid colourings as follows:

(*a*) $0^{t-2}1^{t^2-3t+2}0^{t-2}\left(2^{t^2-3t+2}\{0,2\}^{t-2}\right)^{t-2}2^{t^2-3t+2}0^{t-2}1^{t^2-3t+2}0^{t-2}$,

In colour 0, we have the monochromatic solution

$$1\cdot(t-2)+(t^3-t^2+1)=t^3-t^2-t-1.$$

Note that the integers having colour 1 are $[t-1,t^2-2t]$ (say A), and $[t^3-2t^2+t-1,t^3-t^2-2t]$ (say B). Hence, in colour 1, we have a monochromatic solution

$$(t-1)\cdot(t-2)+(t^3-2t^2+2t-3)=t^3-t^2-t-1,$$

using the fact that $t-1=\min(A)$ and since for $t\geqslant 4$,

$$\begin{aligned}\min(B)=t^3-2t^2+t-1 &< t^3-2t^2+2t-3\\ &= t^3-t^2-2t-(t^2-4t+3)\\ &< t^3-t^2-2t=\max(B).\end{aligned}$$

Note that for $i=0,1,\ldots,t-3$, the integers that are coloured 2 are $[(i+1)(t^2-2t)+t-1,(i+2)(t^2-2t)]$ (say A_i). Then we have a monochromatic solution

$$(t-2)\cdot(t^2-t-1)+(2t^2-2t-3)=t^3-t^2-t-1,$$

using the fact that $t^2-t-1=\min(A_0)$ and since for $t\geqslant 4$,

$$\min(A_1)=2t^2-3t-1<2t^2-2t-3<3t^2-6t=\max(A_1).$$

(*b*) $5\cdot 2^{(t-2)^2}$ more colourings obtained from (*a*) using the permutations $(0)(1,2)$, $(0,1)(2)$, $(0,1,2)$, $(0,2,1)$, and $(0,2)(1)$. As in (*a*), these colourings also cannot be extended.

We do not know, in general, if there are valid colourings of $[1,n-1]$ other than the ones discussed above that could be extended.

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