Noncrossing Tree Partitions & Tiling Algebras

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AMS Special Session on Algebraic and Geometric Combinatorics, NDSU

April 17, 2016
Fix $T$ a tree embedded in a disk with exactly its leaves on the boundary and whose interior vertices have degree at least 3.

Goal: Understand the combinatorics and representation theory related to noncrossing tree partitions.
Noncrossing Tree Partitions

Fix $T$ a tree embedded in a disk with exactly its leaves on the boundary and whose interior vertices have degree at least 3.

$$T = \begin{array}{c}
\begin{array}{ccc}
1 & 2 & 3 \\
5 & 6 & 7 \\
9 & 10 & 8
\end{array}
\end{array}$$

A segment $s = (v_0, \ldots, v_t) = [v_0, v_t]$ with $t \geq 1$ is a sequence of interior vertices of $T$ that takes a “sharp” turn at each $v_i$. In particular, interior vertices of $T$ are not segments.

**Example**

The sequences $(1, 2, 3)$, $(5, 6, 2, 3, 4)$, and $(7, 10)$ are segments. The sequence $(4, 7, 10)$ is not a segment.
A **noncrossing tree partition** $\mathbf{B} = (B_1, \ldots, B_k)$ of $T$ is a set partition of the interior vertices of $T$ where

- vertices in $B_i$ can be connected by pairwise nonoverlapping **red admissible curves** (i.e. curves whose endpoints define segments of $T$ and leave their endpoints to the right) and
- red admissible curves connecting vertices of $B_i$ do not cross those connecting vertices of $B_j$ for $i \neq j$.

Let $\text{NCP}(T)$ denote the poset of noncrossing tree partitions ordered by refinement.

Given $\mathbf{B} \in \text{NCP}(T)$, let $\text{Seg}(\mathbf{B})$ be the segments of $T$ defined by $\mathbf{B}$ (for example, $\text{Seg}(\mathbf{B}) = \{[1, 3], [3, 4], [2, 8], [5, 6], [6, 7], [6, 9]\}$).
Noncrossing tree partitions generalize the classical noncrossing set partitions.

\[ \text{NCP}(T) = \begin{array}{c}
\end{array} \]

**Proposition (G.–McConville)**

If the interior vertices of \( T \) have degree exactly 3, then

\[ \#\text{NCP}(T) = \frac{1}{n+1} \binom{2n}{n} \text{ where } n = \#(\text{interior vertices of } T). \]

**Theorem (G.–McConville)**

The poset \( \text{NCP}(T) \) is a lattice.
One also has a notion of **Kreweras complement** on noncrossing tree partitions.

The Kreweras complement of $\mathcal{B}$ is the unique noncrossing tree partition of $T$ such that when drawn using **green admissible curves** one obtains a noncrossing tree on the interior vertices of $T$.

**Theorem (G.–McConville)**

The map $\text{Kr} : \text{NCP}(T) \longrightarrow \text{NCP}(T)$ is a bijection.
Given a noncrossing tree partition and its Kreweras complement \((B, Kr(B))\), one can obtain all other such pairs by local moves.
Representation theory of $\Lambda_T$

Let $k = \overline{k}$. A tree $T$ defines a finite dimensional $k$-algebra, denoted $\Lambda_T = kQ_T/I_T$, called a **tiling algebra** (Parsons-Simoes ’16). The elements of $kQ_T$ are $k$-linear combinations of paths in $Q_T$.

$$T = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
9 \\
10 \\
\end{array} \quad \rightarrow \quad Q_T = \begin{array}{c}
1 \quad \alpha_5 \\
2 \quad \alpha_6 \\
3 \quad \alpha_4 \\
4 \quad \alpha_8 \\
5 \quad \alpha_3 \\
6 \quad \alpha_2 \\
7 \quad \alpha_7 \\
8 \quad \alpha_8 \alpha_7 \\
9 \quad \alpha_5 \alpha_4 \\
10 \quad \alpha_4 \alpha_6, \alpha_6 \alpha_5, \alpha_5 \alpha_4, \alpha_1 \alpha_3, \alpha_2 \alpha_1 \end{array}$$

{vertices of $Q_T$} = \{e : \text{where } e \text{ is an interior edge of } T\}

{arrows of $Q_T$} = \{e_1 \stackrel{\alpha}{\rightarrow} e_2 : \text{where } e_1 \text{ and } e_2 \text{ form a corner of } T\}

$I_T = \langle \alpha_1 \alpha_2 : \alpha_1 \alpha_2 \rangle = \langle \alpha_2 \alpha_1, \alpha_3 \alpha_2, \alpha_1 \alpha_3, \alpha_5 \alpha_4, \alpha_6 \alpha_5, \alpha_4 \alpha_6, \alpha_8 \alpha_7 \rangle$
Proposition (Butler–Ringel ‘87)

The algebra $\Lambda_T$ is a string algebra since

- each vertex of $Q_T$ has at most two arrows starting at it and at most two arrows ending at it and
- for each arrow $\alpha_1$ of $Q_T$ there are at most two arrows $\alpha_2$, $\alpha_3$ such that $\alpha_1 \alpha_2 \not\in I_T$ and $\alpha_3 \alpha_1 \not\in I_T$.

Thus the indecomposable $\Lambda_T$-modules are given by string modules $M(w)$ (i.e. representations of $Q_T$ supported on connected subgraphs that obey the relations from $I_T$).
Proposition (G.–McConville)

The indecomposable $\Lambda_T$-modules are indexed by the segments of $T$.  

$M(w)$

string module

$w$

string

$s_w$

segment
Simple-minded collections

Noncrossing tree partitions arise as configurations of objects in derived categories.

\[ \Lambda \rightsquigarrow D(\Lambda) \text{ the derived category of } \Lambda \]

(Grothendieck, Verdier ‘60s)

a ring a triangulated category

Objects of \( D(\Lambda) \) are cochain complexes of \( \Lambda \)-modules (i.e.

\[
X = \ldots \xrightarrow{d_{x}^{-2}} X^{-1} \xrightarrow{d_{x}^{-1}} X^{0} \xrightarrow{d_{x}^{0}} X^{1} \xrightarrow{d_{x}^{1}} X^{2} \xrightarrow{d_{x}^{2}} \ldots
\]

that satisfies \( d_{x}^{i+1} \circ d_{x}^{i} = 0 \) for each \( i \in \mathbb{Z} \)) defined up to cohomology.

The category \( D(\Lambda) \) also has a **shift functor** [1] : \( D(\Lambda) \to D(\Lambda) \)

where

\[
X[1] = \ldots \xrightarrow{-d_{x}^{-1}} X^{0} \xrightarrow{-d_{x}^{0}} X^{1} \xrightarrow{-d_{x}^{1}} X^{2} \xrightarrow{-d_{x}^{2}} X^{3} \xrightarrow{-d_{x}^{3}} \ldots
\]
**Question**: When do two rings \( \Lambda \) and \( \Gamma \) have equivalent derived categories (as triangulated categories)?

**Theorem (Rickard ‘89)**

Two rings \( \Lambda \) and \( \Gamma \) are derived equivalent if and only if there is a tilting complex \( T \in \mathcal{D}(\Lambda) \) such that \( \Gamma \cong \text{End}_{\mathcal{D}(\Lambda)}(T) \) (as rings).

**Theorem (Rickard ‘02)**

If \( \Lambda \) is a finite dimensional symmetric \( \kappa \)-algebra, then any simple-minded collection \( \{X_1, \ldots, X_n\} \) defines a tilting complex \( T = \bigoplus_{i=1}^n X_i \).

Other examples of simple-minded collections are given by

- a complete set of nonisomorphic simple modules regarded as elements of \( \mathcal{D}^b(\Lambda \text{-mod}) \) and
- **spherical collections** (Seidel-Thomas) in algebraic geometry.
A collection \( \{X_1, \ldots, X_n\} \) of objects of \( D^b(\Lambda\text{-mod}) \) is **simple-minded** if the following hold for any \( i, j \in [n] \):

i) \( \text{Hom}_{D^b(\Lambda\text{-mod})}(X_i, X_j[k]) = 0 \) for any \( k < 0 \),

ii) \( \text{Hom}_{D^b(\Lambda\text{-mod})}(X_i, X_j) = \begin{cases} k & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \),

iii) the smallest triangulated category containing \( X_1, \ldots, X_n \) and closed under taking summands of objects is \( D^b(\Lambda\text{-mod}) \).
Let $2\text{-}\text{smc}(\Lambda_T)$ denote the set simple-minded collections $\{X_1, \ldots, X_n\}$ where $H^k(X_i) = 0$ for any $i \in \{1, \ldots, n\}$ and any $k \neq 0, -1$.

**Theorem (G.–McConville)**

The map $\{(B, Kr(B))\}_{B \in \text{NCP}(T)} \rightarrow 2\text{-}\text{smc}(\Lambda_T)$ given by

$$(B, Kr(B)) \xrightarrow{\theta} \{M(u)[1] : s_u \in \text{Seg}(B) \text{ where } B \in B\} \cup \{M(v) : s_v \in \text{Seg}(B') \text{ where } B' \in Kr(B)\}$$

is a bijection. Furthermore, this map is compatible with mutations (as introduced by Koenig–Yang ‘13).
Thanks!

\[ \mu^+_{[2,4]} \]  
\[ \mu^-_{[2,4]} \]  
\[ \mu^+_{[6,7]} \]  
\[ \mu^-_{[6,7]} \]  
\[ \mu^+_{[7,10]} \]  
\[ \mu^-_{[7,10]} \]