Minimal length maximal green sequences

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Introduction

Maximal green sequences appear in many areas including
i) representation theory (Keller 2011, Brüstle–Dupont–Pérotin 2013),
ii) cluster algebras (Gross–Hacking–Keel–Kontsevich 2014),
iii) BPS spectra in string theory
(Alim–Cecotti–Cordova–Espahbodi–Rastogi–Vafa 2013)

Quiver mutation

_ Q – a 2-acyclic quiver (i.e., Q has no loops or 2-cycles). Add frozen vertices to Q._

\[ Q = \begin{array}{ccc}
1 & 2 & 3 \\
\ell_1 & \ell_2 & \ell_3 \\
\end{array} \]

_framed quiver

\[ Q = \begin{array}{ccc}
1 & 2' & 3' \\
\ell_1 & \ell_2 & \ell_3 \\
\end{array} \]

_coroframed quiver

Mutate \( Q \) at any non-frozen vertex \( k \) to obtain a quiver \( \mu_k Q \). The quiver \( \mu_k Q \) is obtained from \( Q \) by
(i) inserting a new arrow \( i \to j \) for each 2-path \( i \to k \to j \) in \( Q \)
(ii) reversing arrows incident to \( k \)
(iii) delete any 2-cycles

Combinatorial properties of maximal green sequences

A maximal green sequence of \( Q \) is a sequence \( i = (i_1, \ldots, i_k) \) of non-frozen vertices of \( Q \) where
(i) for all \( j \in [k] \) vertex \( i_j \) is _green_ in \( \mu_{i_{j-1}} \cdots \mu_{i_1} Q \) and
(ii) all vertices in \( \mu_{i_k} \cdots \mu_{i_1} Q \) are _red_.

\( \ell_{\min}(Q) \) – length of shortest maximal green sequence of \( Q \)
\( \ell_{\max}(Q) \) – length of longest maximal green sequence of \( Q \)

If \( Q \) is Dynkin, \( \ell_{\min}(Q) = |Q| \) and \( \ell_{\max}(Q) = |Q| + 1 \) where \( |Q| = n \).
(Brüstle–Dupont–Pérotin 2013)

If \( Q \) is acyclic, \( \ell_{\min}(Q) = |Q| \). (Brüstle–Dupont–Pérotin 2013)

If \( Q \) is mutation type \( A_n \), then \( \ell_{\min}(Q) = |Q| + 3 \cdot \text{cycles of } Q \)

Quivers of mutation type \( A_n \) are adjacency quivers of triangulations of convex \((n + 3)\)-gons.

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Let \( Q \) be a quiver composed of full connected subquivers \( Q_1, Q_2, \ldots, Q_l \) satisfying the following:
\( Q_i \cap Q_j = \{ x \} \)
\( Q_i \cap Q_{i+1} = \{ x \} \) or \( \emptyset \) otherwise
\( x \in Q_i \) if \( x \in Q_i \) then the other is in \( Q_i \).

for every \( i \) the quiver \( Q_i \) is of mutation type \( A_n \).

\[ Q = \begin{array}{ccc}
1 & 2 & 3 \\
\ell_1 & \ell_2 & \ell_3 \\
\end{array} \]

Theorem (G.–McConville–Serhiyenko 2017)

\( \ell_{\min}(Q) = \ell_{\min}(Q_i) - k + \sum_{i=1}^{k} (|Q_1| + |3 \text{-cycles of } Q_i|) \)

Types \( D \) and \( A \)

The theorem applies to quivers of mutation types \( D \) (Vatne 2008) and \( A \) (Bastian 2009). There are four families of mutation type \( D \) quivers.

Families
1. Type I quivers
2. Type II quivers
3. Type III quivers
4. Type IV quivers

Mutation type \( A \) and Type IV quivers have the same underlying graphs.

Corollary (G.–McConville–Serhiyenko 2017)
1. \( \ell_{\min}(Q) = n + 3 \cdot \text{cycles of } Q \)
2. \( \ell_{\max}(Q) = n + 3 \cdot \text{cycles of } Q \)
3. \( \ell_{\min}(Q) = n + 2 \cdot |Q| \)
4. \( \ell_{\min}(Q) = n + 2 \cdot |Q| \)

Idea of proof for Type IV quivers

- Use the Theorem to reduce to calculating the \( \ell_{\min}(Q) \).
- Show that the reduced Type IV quivers have \( \ell_{\min}(Q) = n + 2 \cdot |Q| \).
- These quivers arise from triangulations of a punctured disk.

One keeps track of red and green by adding a lamination to the triangulation.
(Fomin–Thurston 2012)

We construct a maximal green sequence \( i = i_1, i_2, i_3, i_4, i_5 \) of the desired length using the surface model.

Connection to derived equivalence

Question (G.–McConville–Serhiyenko 2017)
If \( Q^1 \) and \( Q^2 \) have derived-equivalent Jacobian algebras \( kQ^1, kQ^2 \) and \( kQ^1, kQ^2 \) are derived-equivalent if and only if \( Q^1 \sim Q^2 \) and \( |3 \text{-cycles of } Q^1| = |3 \text{-cycles of } Q^2| \).

- (mutation type \( A_n \)) If \( kQ^1, kQ^2 \) are derived-equivalent, then \( kQ^1 \sim kQ^2 \) and \( |3 \text{-cycles of } Q^1| = |3 \text{-cycles of } Q^2| \).
- (mutation type \( D \)) There are six conjectural derived equivalence classes. A quiver can be put into one of these forms using mutations that preserve \( \ell_{\min}(Q) \).
(Bastian–Holm–Ladkani 2010)