Minimal length maximal green sequences

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arXiv: 1702.07313

Maurice Auslander Distinguished Lectures and International Conference

April 27, 2017
- Maximal green sequences
- Main result and application
- Techniques for proving these
\( Q \) – a \( 2 \)-acyclic quiver (i.e., \( Q \) has no loops or 2-cycles).
Add **frozen vertices** to \( Q \).

\[
Q = \begin{array}{ccc}
1 & \rightarrow & 2 \\
\leftarrow & \rightarrow & \leftarrow \\
3 & \rightarrow & 1
\end{array}
\]

\[
\tilde{Q} = \begin{array}{ccc}
1' & \rightarrow & 2' \\
\leftarrow & \rightarrow & \leftarrow \\
3' & \rightarrow & 1'
\end{array}
\]

framed quiver

\[
\tilde{Q} = \begin{array}{ccc}
1' & \rightarrow & 2' \\
\leftarrow & \rightarrow & \leftarrow \\
3' & \rightarrow & 1'
\end{array}
\]

coframed quiver

We **mutate** \( \tilde{Q} \) at any non-frozen vertex \( k \) to obtain a quiver \( \mu_k(\tilde{Q}) \).
The quiver \( \mu_k(\tilde{Q}) \) is obtained from \( \tilde{Q} \) by

(i) inserting new arrow \( i \rightarrow j \) for each 2-path \( i \rightarrow k \rightarrow j \) in \( \tilde{Q} \)

(ii) reversing all arrows incident to \( k \)

(ii) delete any 2-cycles
**Definition (Keller, 2011)**

A maximal green sequence of $Q$ is a sequence $i = (i_1, \ldots, i_k)$ of non-frozen vertices of $\hat{Q}$ where

(i) for all $j \in [k]$ vertex $i_j$ is **green** in $\mu_{i_{j-1}} \circ \cdots \circ \mu_{i_1}(\hat{Q})$ and

(ii) all vertices in $\mu_{i_k} \circ \cdots \circ \mu_{i_1}(\hat{Q})$ are **red**.

(1,2) and (2,1,2) are the only maximal green sequences of $Q = 1 \to 2$

← the **oriented exchange graph** of $Q = 1 \to 2$
Maximal green sequences can be identified with

- finite length maximal chains in the poset of functorially finite torsion classes of the **Jacobian algebra** of $Q$, [Brüstle–Yang, 2014]
- certain sequences of reachable chambers in the **consistent scattering diagram** of $Q$ [Gross–Hacking–Keel–Kontsevich, 2014]

$Q = 1 \rightarrow 2 \rightarrow 3$  
$Q$ oriented 3-cycle
**Goal:** Understand combinatorial properties of the maximal green sequences of $Q$.

**Conjecture ("No Gap Conjecture" Brüstle–Dupont–Pérotin, 2013)**

For each $\ell_{\text{min}}(Q) \leq k \leq \ell_{\text{max}}(Q)$, there exists a maximal green sequence of length $k$ where

- $\ell_{\text{min}}(Q) := \text{length of shortest maximal green sequence of } Q$
- $\ell_{\text{max}}(Q) := \text{length of longest maximal green sequence of } Q$

- True in mutation type $\mathbb{A}$ [G.–McConville, 2015]
- True for tame hereditary algebras [Hermes–Igusa, 2016]
\( \ell_{\text{min}}(Q) := \text{length of shortest maximal green sequence of } Q \)
\( \ell_{\text{max}}(Q) := \text{length of longest maximal green sequence of } Q. \)

- If \( Q \) is Dynkin, \( \ell_{\text{min}}(Q) = |Q_0| \) and \( \ell_{\text{max}}(Q) = |\Phi^+(Q)|. \) [Brüstle–Dupont–Pérotin, 2013]
- If \( Q \) is acyclic, \( \ell_{\text{min}}(Q) = |Q_0|. \) [Brüstle–Dupont–Pérotin, 2013]
- If \( Q \) is mutation type \( A \), then \( \ell_{\text{min}}(Q) = |Q_0| + \{|3\text{-cycles of } Q\}| \) [Cormier–Dillery–Resh–Serhiyenko–Whelan, 2015]
Let $\widetilde{Q}$ be a quiver composed of full connected subquivers $Q, Q^1, Q^2, \ldots, Q^k$, such that all of the following conditions hold.

- $Q^i_0 \cap Q_0 = \{x_i\}$.
- $Q^i_0 \cap Q^j_0 = \begin{cases} \{x_i\} & \text{if } x_i = x_j \\ \emptyset & \text{otherwise} \end{cases}$.
- If $\alpha \in \widetilde{Q}_1$ has an endpoint in $Q^i_0 \setminus \{x_i\}$ then the other is in $Q^i_0$.
- For every $i$ the quiver $Q^i$ is of mutation type $A$.

\[ \ell_{\min}(\widetilde{Q}) = \ell_{\min}(Q) - k + \sum_{i=1}^{k} (|Q^i_0| + |\{3\text{-cycles in } Q^i\}|) \]
The theorem applies to quivers of mutation types $\mathbb{D}$ [Vatne, 2008] and $\tilde{\mathbb{A}}$ [Bastian, 2009]. There are four families mutation type $\mathbb{D}$ quivers.

**Figure: Type I quivers**

**Figure: Type II quivers**

**Figure: Type III quivers**

**Figure: Type IV quivers**

Mutation type $\tilde{\mathbb{A}}$ and Type IV quivers have the same underlying graphs.
Corollary (G.–McConville–Serhiyenko, 2017)

i) \( \ell_{\text{min}} = n + |\{3\text{-cycles in } \tilde{Q}\}| \) (\( \tilde{Q} \) is of Type I or of type \( \tilde{A}_{n-1} \))

ii) \( \ell_{\text{min}} = n + 1 + |\{3\text{-cycles in } Q^1\}| + |\{3\text{-cycles in } Q^2\}| \) (\( \tilde{Q} \) is of Type II)

iii) \( \ell_{\text{min}} = n + 2 + |\{3\text{-cycles in } \tilde{Q}\}| \) (\( \tilde{Q} \) is of Type III)

iv) \( \ell_{\text{min}} = n + k - 2 + |\{a_i : \deg(a_i) = 4\}| + \sum_{i=1}^{k} |\{3\text{-cycles in } Q^i\}|. \)
If $Q^1$ and $Q^2$ have derived-equivalent cluster-tilted algebras $\mathbb{k}Q^1/I^1$ and $\mathbb{k}Q^2/I^2$, is $\ell_{\min}(Q^1) = \ell_{\min}(Q^2)$?

- (mutation type $\mathbb{A}$) If $\mathbb{k}Q^1/I^1$ and $\mathbb{k}Q^2/I^2$ are derived-equivalent if and only if $|Q^1_0| = |Q^2_0|$ and $|\{3\text{-cycles of } Q^1\}| = |\{3\text{-cycles of } Q^2\}|$. [Buan–Vatne, 2007]

- (mutation type $\tilde{\mathbb{A}}$) If $\mathbb{k}Q^1/I^1$ and $\mathbb{k}Q^2/I^2$ are derived-equivalent, then $|Q^1_0| = |Q^2_0|$ and $|\{3\text{-cycles of } Q^1\}| = |\{3\text{-cycles of } Q^2\}|$. [Bastian, 2009]

- (mutation type $\mathbb{D}$) There are six conjectural derived equivalence classes. A quiver can be put into one of these forms using mutations that preserve $\ell_{\min}(Q)$. [Bastian–Holm–Ladkani, 2010]
To prove the theorem:

- construct a maximal green sequence of length
  \[ \ell_{\min}(Q) - k + \sum_{i=1}^{k} (|Q_0^i| + |\{3-cycles in Q^i\}|) \]
- show there are no shorter maximal green sequences (*)

To address *, one uses the \textbf{c-vectors} of \( Q \). These record the arrows between non-frozen vertices and frozen vertices.
We identify maximal green sequences with their sequences of c-vectors.

\[ \mathbf{i} = (i_1, \ldots, i_k) \longleftrightarrow \mathbf{c}(\mathbf{i}) = (c_1, \ldots, c_k) \]

The following was essentially proved by [Muller, 2015].

**Theorem (G.–McConville–Serhiyenko, 2017)**

Let \( Q \) be a 2-acyclic quiver and \( Q^\dagger \) any full subquiver. There is a map \( \text{MGS}(Q) \to \text{MGS}(Q^\dagger) \) sending \( \mathbf{i} \in \text{MGS}(Q) \) to \( \mathbf{i}^\dagger \in \text{MGS}(Q^\dagger) \) where \( \mathbf{c}(\mathbf{i}^\dagger) = (c_1^{(1)}, \ldots, c_\ell^{(\ell)}) \) is the subsequence of \( \mathbf{c}(\mathbf{i}) \) where each \( c_j^{(i)} = (c_1^{(j)}, \ldots, c_n^{(j)}) \) satisfies \( c_i^{(j)} = 0 \) if \( i \in Q_0 \setminus Q_0^\dagger \).

(The proof uses properties of the consistent scattering diagram of \( Q \).)

\[ \implies \text{ Let } \mathbf{i} \in \text{MGS}(\tilde{Q}). \]

\[
\ell(\mathbf{i}) \geq \ell(\mathbf{i}_{(\tilde{Q}_0 \setminus Q_0^\dagger) \cup \{x_1\}}) + \ell(\mathbf{i}_{Q_0^\dagger}) - 1 \\
\geq \ell(\mathbf{i}_{(\tilde{Q}_0 \setminus Q_0^1 \cup Q_0^2) \cup \{x_1, x_2\}}) + \ell(\mathbf{i}_{Q_0^1}) + \ell(\mathbf{i}_{Q_0^2}) - 2 \\
\vdots \\
\geq \ell_{\text{min}}(Q) - k + \sum_{i=1}^{k} \ell_{\text{min}}(Q^i) 
\]
To prove the corollary, use the Theorem to reduce to calculating the $\ell_{\text{min}}(Q)$. We focus on the mutation type $\mathbb{D}$ case.

Figure: $\ell_{\text{min}}(Q) = 3$

Figure: $\ell_{\text{min}}(Q) = 4 + 1$

Figure: $\ell_{\text{min}}(Q) = 4 + 2$

Figure: $\ell_{\text{min}}(Q) = n + k - 2 + |\{a_i : \deg(a_i) = 4\}|$
Show that the reduced Type IV quivers have
\[ \ell_{\min}(Q) = n + k - 2 + |\{a_i : \deg(a_i) = 4\}|. \] (not quite easy)

- These quivers arise from triangulations of a punctured disk.
When $Q$ is defined by a triangulation, one keeps track of red and green by adding a **lamination** to the triangulation. [Fomin–Thurston, 2012]
Show that the reduced Type IV quivers have
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- One keeps track of red and green by adding a **lamination** to the triangulation. [Fomin–Thurston, 2012]

- We construct a maximal green sequence \( i = i_1 \circ i_2 \circ i_3 \circ i_4 \circ i_5 \) of the desired length. (*)
Thanks!