Maximal Green Sequences and Type $\Delta$ Quivers
Alexander Garver and Gregg Musiker
University of Minnesota

Introduction
Maximal green sequences appear in many areas including
• i) quantum dilogarithm identities in representation theory
• ii) Cambrian lattices in combinatorics
• iii) computations of BPS spectra in physics.
We construct a maximal green sequence for each quiver mutation equivalent to an oriented type $\Delta$ Dynkin diagram.

Notation
Given a quiver $Q$, the framed quiver (resp. coframed quiver) of $Q$, denoted $\widehat{Q}$ (resp. $\check{Q}$), is formed by
(i) adding a frozen vertex $i$ for each vertex $i$ in $Q$
(ii) adding an arrow $i \rightarrow j$ (resp. $i \leftarrow j$) for each vertex $i$ in $Q$.
A vertex $i$ of $\overline{Q} = \text{Mut}(\check{Q})$ is green (resp. red) if all arrows frozen vertices and $i$ point away from (resp. toward) $i$.

\[ Q = \begin{array}{c}
1
\end{array} \]
\[ \overline{Q} = \begin{array}{c}
1' & 2'
\end{array} \]

Theorem (DWZ, “Sign Coherence of c-vectors”)
Each vertex $i$ of $\overline{Q} = \text{Mut}(\check{Q})$ is either green or red, but not both.

Definition
A mutation sequence $\mu = \mu_{i_1} \cdots \mu_{i_k}$ is a maximal green sequence if
(i) for all $j \in [k]$ vertex $i_j$ is green in $\mu_{i_1} \cdots \mu_{i_j} \check{Q}$
(ii) all vertices in $\mu \check{Q}$ are red.

Example
The quiver $Q$ has maximal green sequences $\mu_2 = \mu_1$ and
$\mu_2 = \mu_1 + \mu_2$.

\[ \overline{Q} = \begin{array}{c}
1' & 2'
\end{array} \]

Theorem (Brüstle-Dupont-Pérotin)
If $Q$ is an acyclic orientation of a simply-laced Dynkin diagram or affine Dynkin diagram, then green $Q$, the set of maximal green sequences of $Q$, is non-empty and finite.

\[ \text{Definition} \]
Let $Q_1, Q_2$ be quivers. Let $\{a_1, \ldots, a_l\}$ be an $k$-multiset on $Q_1$ and $\{b_1, \ldots, b_l\}$ be a $k$-multiset on $Q_2$. Define the direct sum of $Q_1$ and $Q_2$, $Q_1 \oplus \{b_1, \ldots, b_l\} Q_2$, to be the quiver with vertices
\[ (Q_1 \oplus \{b_1, \ldots, b_l\} Q_2)_v = (Q_1)_v \cup (Q_2)_v \cup \{a_1, \ldots, a_l\} \subseteq v \in |Q| \}
and arrows
\[ (Q_1 \oplus \{b_1, \ldots, b_l\} Q_2)_a = (Q_1)_a \cup (Q_2)_a. \]
We say a quiver $Q$ is irreducible if $Q = Q_1 \oplus \{b_1, \ldots, b_l\} Q_2$ implies that $Q_1 = \varnothing$ or $Q_2 = \varnothing$.

Example
\[ Q_1 \oplus \{5,8,11,9,11\} Q_2 = \begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{array} \]

Proposition (G. - Musiker)
Let $Q = Q_1 \oplus \{b_1, \ldots, b_l\} Q_2$, where $\#(\{a_i, \ldots, b_j\} < 1$ for all $i, j \in [k]$. If $\mu_q = \text{green}(Q_1)$ and $\mu_u = \text{green}(Q_2)$, then $\mu_q + \mu_u = \text{green}(Q_l)

Lemma
Let $N \geq 3$. The irreducible type $\Delta_N$ quivers are exactly those quivers $Q$ that can be obtained from a collection of 3-cycles $\{T_i\}$ by identifying their vertices in such a way that the following are satisfied.
(i) any $i, j \in Q$ satisfy $i \neq j$ < 1
(ii) each $i \in Q$ is identified with at most one other vertex in $Q_0$
(iii) each cycle in the underlying graph of $Q$ is induced by a $T$.

Example
We can label any irreducible type $\Delta$ quiver $Q$ as follows.
\[ Q = \begin{array}{c}
\end{array} \]

Definition
Let $Q$ be an embedded, irreducible type $\Delta$ quiver. Define the associated mutation sequence of $Q$ by $\mu = \mu_1 + \cdots + \mu_{|Q|}$ $(n \geq 3$-cycles in $Q)$ where $\mu_i = \mu_i$ and $\mu_1 = \mu_{|Q|}$

Example
Let $\overline{Q}$ be an embedded, irreducible type $\Delta$ quiver. Define the combinatorial DT-invariant of $Q$. Since $\mu \in \text{green} Q$, we have $\overline{Q} = \overline{\mu} = \overline{\mu}', \cdots \overline{\mu}'$ where each $\overline{\mu}'$ is a product of quantum dilogarithm series.