Rook placements on South-West permutation diagrams

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Abstract

We study rook placements over South-West diagrams of permutations of given length. Our main motivation comes from an article from Foata and Schützenberger, [3], and a serie of papers published in the 70s by Goldman, Joichi and White about Rook Theory, [5, 7]. We talk about rook-equivalence in $\mathfrak{S}_n$ and give some results about equivalence classes. We also propose a labelling of Ferrers board with many interesting properties. Finally, we examine some q-analogues for this problem.

Résumé

On étudie les placements de tours sur les diagrammes Sud-Ouest de permutations. Notre motivation principale provient d'un article de Foata et Schützenberger, [3], ainsi que d'une série d'articles publiées dans les années 70 par Goldman, Joichi et White, au sujet de la théorie des placements de tours, [5, 7]. On aborde la relation d'équivalence induite par les placements de tours dans $\mathfrak{S}_n$, et on donne quelques résultats au sujet des classes équivalences. On propose ensuite une fonction d'étiquettage sur les diagrammes de Ferrers, dont on exhibe quelques propriétés intéressantes. Finalement, on examine brièvement la question des q-analogues pour les placements de tours.

1 Introduction

The main problem is to tell when two given chess boards are rook-equivalent, that is when the number of ways of placing k nontaking rooks are the same on both boards for all k. Foata and Schützenberger in [3], and later Goldman, Joichi and White in [5], made a lot of progress by giving a complete answer for the particular case of Ferrers boards. Goldman, Joichi and White also unveiled in [7] a link between rook-equivalence and graph coloration.

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Our goal is to extend the complete classification of rook-equivalence over Ferrers boards to a larger class that contains Ferrers board, called permutation diagrams. These diagrams are related, among others, to Schubert calculus (see [9]). They have been studied in many papers, notably in [1] by Morales and Lewis, and in [8] by the same authors along with Klein.

Section 3 is mainly devoted to an "extension operation", in which we adapt most of the existing results concerning Ferrers boards to the new framework of permutation diagrams. We try to generalize the statements whenever possible. We check how the Ferrers part interact with other permutation diagrams. With existing tools, we also give a very precise condition for the membership of a permutation to what we call dominant classes: a permutation is rook-equivalent to a dominant permutation (that is, permutation whose diagram is a Ferrers board) if and only if its rook polynomial has only integer roots.

Section 4 adresses the study of rook-equivalence classes in $\mathfrak{S}_n$. We start by showing that there are 2^{n-1} classes in $\mathfrak{S}_n$ containing a dominant permutation. We then study a type of classes we call "mixed" (composed of both covexillary and non-covexillary permutations), about which we give a conjecture (Conjecture 2). As an evidence of this conjecture, we give a method that allows to construct an infinite collection of permutations for which the conjecture is proved. More precisely, we show that the following permutation is a non-covexillary member of a mixed class:

$$\omega = (\rho_{2m+2} \oplus \rho_1) \ominus (\rho_m \oplus \rho_1).$$

The 5th section is devoted to a labelling of Ferrers boards with many interesting properties, mainly related to the work of Foata, Schützenberger, Goldman, Joichi and White. The labelling was inspired by a paper of Ding, [2]. Our labelling is a function from B to $\mathbb{N}$ defined by

$$\Phi_B(i,j) = n + i - j.$$

Finally we address in Section 6 the question of q-analogues for rook placements, mainly studied by Garsia, Remmel, Haglund, and many others. Using a q-analogue defined by Garsia and Remmel, we give a conjecture concerning the new equivalence relation induced by q rook numbers.

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2 Preliminaries

2.1 Boards and South-West diagrams

We use the following convention through this document: the integer interval $\{i \in \mathbb{N} : a \leqslant i \leqslant b\}$ will be noted $[a, b]$. If $a = 1$, we simply note $[b]$ in place of $[1, b]$.

We call a *board* a subset B of $[n] \times [n]$. A board $B \subseteq [n] \times [n]$ is called *proper* if it satisfies:

1. for all $(i, j) \in B$, $i < j$;
2. B defines a transitive relation on $[n]$, that is $(i, j) \in B$ and $(j, k) \in B$ implies $(i, k) \in B$.

Defining $T_n = \{(i, j) \in [n] \times [n] : i < j\}$, it follows that any transitive relation B on $[n]$ is proper as soon as $B \subseteq T_n$.

Moreover, if a proper board B has the property that whenever $t \in B$, all tiles North or East of t are also in B, then B will be called a *Ferrers board.* The tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$, where λ_i is the number of boxes in the ith row of B, is called the *shape* of B. By definition, λ is always decreasing, and if λ is *strictly* decreasing except for its zero entries, we say that B is a *decreasing Ferrers board.* Remark that because λ is a partition, this definition indeed corresponds to the classical definition of Ferrers diagrams, up to the fact that our Ferrers boards will be right-aligned, decreasing, and strictly upper triangular. These conventions will be useful when dealing with permutation diagrams.

Given a permutation σ of length n (written in one-line notation), we define the following n-board to be the *South-West diagram* of σ:

$$D_\sigma = \{(i, \sigma(j)) : i < j, \sigma(i) < \sigma(j)\}.$$

Such a board can be obtained by placing n points $(i, \sigma(i))$ on $[n] \times [n]$ (using matrix notation) and crossing all tiles South and West of those points, hence the name South-West diagram. Figure 1 gives an example.

Some may see the resemblance between South-West diagrams and Rothe diagrams (which are in fact South-East diagrams). As we will exclusively be concerned with the former, we will allow ourselves to recycle some of the terms usually defined for Rothe diagrams. For instance, a permutation σ will be referred to as dominant if D_σ has the shape of a partition, that is if D_σ is in fact a Ferrers board, as defined above. The reason why D_σ should necessarily be strictly upper triangular will be given by Lemma 1.

Example 1. *The permutation* $(2\,3\,4\,5\,1)$ *is dominant, and the shape of its diagram is* $(3, 2, 1, 0, 0)$*. On the other hand, the permutation* $(5\,1\,2\,3\,4)$ *is not dominant, even though the shape of its diagram appears to a partition: in fact, its shape is* $(0, 3, 2, 1, 0)$ *which is not a partition. See Figure 1 for illustrations.*

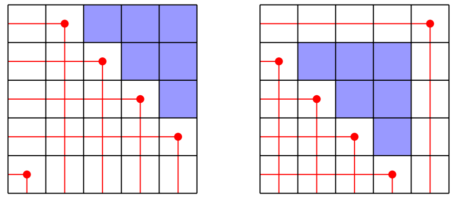

Figure 1: The South-West diagrams of (2 3 4 5 1) and (5 1 2 3 4). The leftmost diagram is dominant, whereas the other is not.

We will say that a permutation σ is *covexillary* if the rows of D_σ (and, consequently its columns) can be totally ordered by inclusion, that is if for every i, j we either have $(i,k) \in D_\sigma$ whenever $(j,k) \in D_\sigma$ or the converse. It is well known that covexillary permutations are exactly those for which there exists no subsequence order-isomorphic to (3 4 1 2). We will say that such a permutation *avoids the pattern* (3 4 1 2). Note that every dominant permutation is covexillary, because its rows are naturally ordered by inclusion from bottom to top.

Example 2. *The permutation* (5 8 1 6 3 7 2 4) *is non-covexillary because, amongst others, the subsequence* $(5, 8, 1, 3)$ *is order-isomorphic to* (3 4 1 2). *See Figure 2 for an illustration.*

The code of σ, noted c_σ, is a tuple of length n in which the ith element is the number of points (i,k) in D_σ. It can be shown (see [9], chapter 2) that a permutation σ is dominant if and only if c_σ is a partition (that is, decreasing). For instance, the permutation $\sigma = (2\ 3\ 4\ 5\ 1)$ is dominant and its code is $c_\sigma = (3, 2, 1, 0, 0)$. If c_σ is *strictly* decreasing (with the exception of zero entries), then σ will be referred to as strictly dominant.

A more thourough treatement of these notions (though defined for Rothe diagrams instead) may be found in a book written by Manivel, *Fonctions Symétriques, polynômes de Schubert et lieux de dégénérescence* [9].

2.2 Rook placements

Rook Theory mainly consists in counting the number of ways of placing non-taking rooks on arbitrary boards. Define for a board B the set of all k rooks placements on B:

$$R_k(B) = \{R \subseteq B : \#R = k,\ \text{no two } (i,j) \in R \text{ sharing a coordinate}\}.$$

We will call an element R of $R_k(B)$ a k rooks placement over B, whereas an element (i,j) of R will be called a rook. We conveniently represent a rooks placement $R \in R_k(B)$ by placing k nontaking rooks on B.

Similarly, we define the set of all k-hit rooks placements over a board B:

$$H_k(B) = \{\tau \in \mathfrak{S}_n : \#\{i : (i, \tau_i) \in B\} = k\}.$$

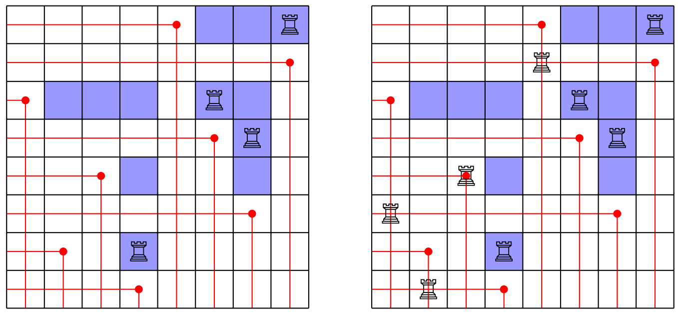

Figure 2: A 4-rook placement and 4-hit rook placement on the South-West diagram of $(5\,8\,1\,6\,3\,7\,2\,4)$.

$H_k(B)$ is the set of all n-rook placements over $[n] \times [n]$ (elements of $\mathfrak{S}_n$) for which k rooks lie on B. An element $H \in H_k(B)$ is called a k hit placements. We give an example of a k rooks placement and k hit placement in Figure 2.

We will write $r_k(B) := \#R_k(B)$ and $h_k(B) := \#H_k(B)$. Furthermore, if $B = D_\sigma$ for a certain σ, we will write $R_k(\sigma)$ in place of $R_k(D_\sigma)$, and so on, without ambiguity.

Finally, we will be interested in the relation of rook equivalence between two n-board B, C, which occurs if $r_k(B) = r_k(C)$ for all $k \geqslant 0$. We will simply write $B \sim C$ and $\sigma \sim \tau$ for two permutations $\sigma, \tau \in \mathfrak{S}_n$ if $D_\sigma \sim D_\tau$.

2.3 Multiset and tuples

We give some basic definitions about tuples and multisets, without too much formalism.

A tuple $T = (t_1, \ldots, t_k)$ is an ordered collection of elements (not necessarily distincts). If $U = (u_1, \ldots, u_l)$ is another tuple, the concatenation of T and U is the tuple

$$TU = (t_1, \ldots, t_k, u_1, \ldots, u_l)$$

We write the concatenation of a finite family of tuples $T_1, \ldots, T_n$

$$\prod_{i=1}^{n} T_i = T_1 \ldots T_n.$$

T^n represents the concatenation $\prod_{i \in [n]} T$. If $n = 0$, then $T^0 = \epsilon$, where ϵ is the empty tuple.

Moreover, the indicator of T, noted $\mathbf{1}_T$ is a function from T to $\{0, 1\}$ defined by

$$\mathbf{1}_T(t) = \begin{cases} 1 & \text{if } t \in T \\ 0 & \text{otherwise} \end{cases}$$

For the notion of multiset, we mainly use the definition of Stanley in [11]. Let A be a set. A multiset M on A is a function $\nu : A \to \mathbb{N}$ satisfying

$$\sum_{a \in A} \nu(a) < \infty$$

The sum $\sum_{a \in A} \nu(a)$ is called the cardinality of M, noted $\#M$, and M itself will be noted

$$M = \{(a, \nu(a)) : a \in A\}$$

3 Permutation diagrams

3.1 Dominant permutations and Ferrers boards

This subsection is mainly technical. It aims to clarify the definition of dominant permutation and to show the relevance of our unusual definition of Ferrers board. We start by a very simple lemma coming from the book of Laurent Manivel, *Fonctions symétriques, polynômes de Schubert et lieux de dégénérescence* [9]:

Lemma 1. *Let $\sigma \in \mathfrak{S}_n$ and $c_\sigma = (c_1, \ldots, c_n)$. Then for all $c_i \in c_\sigma$,*

$$c_i \leqslant n - i$$

Proof. Recall that

$$D_\sigma = \{(i, \sigma(j)) : i < j, \sigma(i) < \sigma(j)\}.$$

For a fixed i, the set $\{j : i < j \leqslant n\}$ has $n - i$ elements, and thus there can be no more than $n - i$ elements in D_σ with the first coordinate being i. □

This lemma explains why we defined our Ferrers board to be strictly upper triangular: the diagram of a partition that is not strictly upper triangular can not be the diagram of a permutation.

We now give an explicit formula to compute the dominant permutation corresponding to a given Ferrers board:

Proposition 1. *Let $B \subseteq T_n$ be a Ferrers board of shape $\lambda = (\lambda_1, \ldots, \lambda_n)$. Define recursively the sequence of sets $\{A_i\}$ for $i \in [n]$:*

$$A_i = \{j \in [n - \lambda_i] : \forall k \in [i-1], j \neq \max(A_k)\}.$$

Define the map σ by $\sigma(i) = \max(A_i)$, $i \in [n]$. Then $\sigma \in \mathfrak{S}_n$ and $D_\sigma = B$.

Proof. The proof is in two parts: we first show that $\sigma \in \mathfrak{S}_n$, then we show that $D_\sigma = B$.

Because $B \subseteq T_n$, we have $0 \leqslant \lambda_i \leqslant n - i$, and thus

$$0 \leqslant \lambda_i \leqslant n - i \iff i - n \leqslant -\lambda_i \leqslant 0 \iff i \leqslant n - \lambda_i \leqslant n. \quad (1)$$

As $j \in A_i$ implies $j \in [n - \lambda_i]$, the range of σ is indeed in $[n]$. Furthermore, the set $\{\max(A_j) : j \in [i-1]\}$ has $i-1$ elements, whereas, by equation 1, $[n-\lambda_i]$ has at least i elements. Therefore $A_i \neq \varnothing$. Thus the maximum of A_i exists for all $i \in [n]$ and σ is well defined as a map from $[n]$ onto itself.

Take $l, m \in [n]$ and asssume $l < m$. Because $\sigma(m) = \max(A_m)$ by definition, we have $\sigma(m) \in A_m$. By construction of A_m, $\sigma(k) \neq \sigma(m)$ whenever $k < m$. In particular, we must have $\sigma(l) \neq \sigma(m)$ as we assumed $l < m$. It shows that $\sigma : [n] \hookrightarrow [n]$, that is σ is an injection from $[n]$ to $[n]$. Because its range is equal its domain, σ is also a bijection and is well defined as a permutation of $\mathfrak{S}_n$, as desired.

To show that $D_\sigma = B$, we start by showing that for a given $i \in [n]$, $(i, \sigma(j)) \in D_\sigma$ if and only if $n - \lambda_i < \sigma(j) \leqslant n$. Recall the definition of D_σ:

$$D_\sigma = \{(i, \sigma(j)) : i < j, \sigma(i) < \sigma(j)\}.$$

Start with the "if" part, and suppose $(i, \sigma(j)) \in D_\sigma$. Then $j > i$ by definition of D_σ. We already know that $\sigma(j) \leqslant n$, and thus we just need to show that $\sigma(j) > n - \lambda_i$. Suppose that $\sigma(i) < \sigma(j) \leqslant n - \lambda_i$. Then $\sigma(j) \in [n - \lambda_i]$, and as $j > i$, $\sigma(j) \neq \sigma(k)$ for all $k \in [i-1]$. Therefore we have $\sigma(j) \in A_i$, but we also have $\sigma(j) > \sigma(i) = \max(A_i)$, which is a contradiction. So we indeed have $n - \lambda_i < \sigma(j) \leqslant n$ whenever $(i, \sigma(j)) \in D_\sigma$.

For the "only if" part, suppose $\sigma(j) > n - \lambda_i$. Because $\sigma(j) \in A_j$, we have $\sigma(j) \leqslant n - \lambda_j$. Therefore,

$$n - \lambda_i < \sigma(j) \leqslant n - \lambda_j \Rightarrow \lambda_j < \lambda_i,$$

and because λ is a partition, it implies that $j > i$. Thus, whenever $n - \lambda_i < \sigma(j) \leqslant n$, we have $(i, \sigma(j)) \in D_\sigma$.

That being said, we can compute the ith element of c_σ:

$$c_i = \#\{j : (i, \sigma(j)) \in D_\sigma\} = \#\{n - \lambda_i < j \leqslant n\} = \lambda_i,$$

as desired. Suppose $(i, j) \in D_\sigma$, we can show that for $k < i$ and $l > j$, both (i, l) and (k, j) are in D_σ. As $(i, j) \in D_\sigma$, $n - \lambda_i < j \leqslant n$, and thus $j < l \leqslant n$ implies $(i, l) \in D_\sigma$. On the other hand, as $k < i$, $n - \lambda_k < n - \lambda_i < j$ and thus $n - \lambda_k < j \leqslant n$, and we have $(k, j) \in D_\sigma$ as desired. □

We find that each Ferrers board is the diagram of dominant permutation, and that each dominant permutation has for diagram a Ferrers board. It is not hard to see also that our Ferrers board are in fact Dyck path in $[n] \times [n]$, and it is well-known that these are counted by Catalan numbers. Thus, we have the following corollary:

Corollary 1. *The number of dominant permutations in $\mathfrak{S}_n$ is equal to the nth Catalan number,*

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

3.2 Skew and direct sum of permutations

We define the skew and direct sum of two permutations as follow:

Definition 1. *Let $\sigma \in \mathfrak{S}_n$, $\tau \in \mathfrak{S}_m$. Then the skew sum $\tau \ominus \sigma$ is*

$$(\tau \ominus \sigma)(i) = \begin{cases} \tau(i) + n & \text{if } 1 \leqslant i \leqslant m, \\ \sigma(i-m) & \text{if } m+1 \leqslant i \leqslant n+m. \end{cases}$$

The direct sum $\tau \oplus \sigma$ is

$$(\tau \oplus \sigma)(i) = \begin{cases} \tau(i) & \text{if } 1 \leqslant i \leqslant m, \\ \sigma(i-m) + m & \text{if } m+1 \leqslant i \leqslant m+n. \end{cases}$$

We give two properties that show how rook-equivalence behaves with skew sum:

Proposition 2. *Let $\sigma \in \mathfrak{S}_n, \tau \in \mathfrak{S}_m$ and let $\rho_k = (\, k \; k{-}1 \; \dots \; 1 \,)$ be the kth order reversing permutation. Then,*

1. $r_k(\sigma \ominus \tau) = \sum_{i+j=k} r_i(\sigma) r_j(\tau)$;
2. $r_k(\sigma \ominus \rho_k) = r_k(\sigma) = r_k(\rho_k \ominus \sigma)$.

The convolution property comes from the fact that the diagram of $\sigma \ominus \tau$ is a disjoint union of D_σ and D_τ. Therefore, placing j rooks on D_σ and then i rooks on D_τ yields an $i+j$ rooks placement on $D_{\sigma \ominus \tau}$. The second property is an immediate consequence of the South-West construction of permutation diagrams.

3.3 Factoring the rook polynomial

The following polynomial, as defined by Goldman, Joichi and White in [5], will be useful for studying rook-equivalence:

Definition 2. *Let $\sigma \in \mathfrak{S}_n$. The rook polynomial of σ is given by*

$$\chi_\sigma(x) = \sum_{k \geqslant 0} r_k(\sigma)(x)_{n-k}.$$

where $(x)_j = x(x-1)\dots(x-j+1)$ refers to the j falling factorial of x.

The next theorem originates from [5], but was expressed in a different notation. We rephrase it in terms of permutation diagrams:

Theorem 1 (Goldman-Joichi-White)**.** *For every dominant permutation $\sigma \in \mathfrak{S}_n$ of code $c_\sigma = (c_1, \dots, c_n)$,*

$$\chi_\sigma(x) = \prod_{i=1}^{n} (x + c_i - n + i).$$

We can adapt this result for all covexillary permutations. First we need a definition and a lemma:

Definition 3. *Let $\sigma \in \mathfrak{S}_n$, we say that the following set is a* row *of D_σ:*

$$l_i = \{k : (i,k) \in D_\sigma\}$$

We say that row i is smaller than row j (written $l_i \leqslant l_j$) either if $l_i = l_j$ as a set and $i \geqslant j$, or if $l_i \subset l_j$. We denote by s_i the number of rows strictly smaller than row i in D_σ.

Remark 1. *For a covexillary permutation σ, the partial order defined above is a strict total order, since by definition the rows of D_σ can be ordered by inclusion. Moreover, for σ dominant, we have $s_i = n - i$ because the rows are ordered from bottom to top.*

We now give a more general form of Theorem 1:

Proposition 3. *For any covexillary permutation $\sigma \in \mathfrak{S}_n$, let $c_\sigma = (c_1, \ldots, c_n)$, then*

$$\chi_\sigma(x) = \prod_{i=1}^{n}(x + c_i - s_i).$$

Proof. We use the same idea Goldman, Joichi and White used to prove Theorem 1 in [5].

Fix $x \in \mathbb{N}$ and let

$$X = [n] \times [n+1, n+x]$$

For $B = D_\sigma \sqcup X$, we show that:

$$\prod_{i=1}^{n}(x + c_i - s_i) = r_n(B) = \sum_{k \geqslant 0} r_k(\sigma)(x)_{n-k}$$

The rows of B can be totally ordered by inclusion because σ is covexillary and the rows of X are all equal as sets. Thus, if l_i is the ith row of B, we can write:

$$l_{i_1} < \cdots < l_{i_n},$$

for some $i_1, \ldots, i_n \in [n]$. Because the choice of a rooks placement $R \in R_n(B)$ is equivalent to a choice of one element in each l_i, no two equal, we can count $r_n(B)$ by counting the choices row by row. We know that $\#l_i = x + c_i$. Starting with l_{i_1}, we have $x + c_{i_1}$ choices, s_{i_1} being zero, as desired. Continuing up to row l_{i_k}, we have $x + c_{i_k} - s_{i_k}$ choices, as we cannot choose one of the s_{i_k} elements already chosen. Thus,

$$r_n(B) = \prod_{k=1}^{n}(x + c_{i_k} - s_{i_k}) = \prod_{i=1}^{n}(x + c_i - s_i)$$

For the second part, let

$$R_n^k(B) = \{R \in R_n(B) : (R \cap D_\sigma) \in R_k(\sigma)\}$$

Because for each $R \in R_n(B)$, the intersection $(R \cap D_\sigma)$ is a k rooks placement over D_σ for some k, we have that

$$R_n(B) = \bigsqcup_{k \geqslant 0} R_n^k(B) \Rightarrow r_n(B) = \sum_{k \geqslant 0} \# R_n^k(B)$$

We can choose $R \in R_n^k(B)$ by first placing k rooks in $r_k(\sigma)$ ways, and then placing $n-k$ rooks in the $n-k$ remaining rows of X. This last operation can be done in $(x)_{n-k}$ ways, as we have x free boxes on the first row, $x-1$ on the second, and so on. Thus,

$$R_n(B) = \sum_{k \geqslant 0} \# R_n^k(B) = \sum_{k \geqslant 0} r_k(\sigma)(x)_{n-k},$$

completing the proof. □

There is a straightforward corollary of Theorem 1: if for a given permutation σ there exists a dominant permutation τ such that $\sigma \sim \tau$, then all the roots of χ_σ are in $\mathbb{N}$. We want to show that the converse is also true. With that, we completely characterize any permutation rook-equivalent to a dominant permutation. First we need a theorem from Goldman, Joichi and White (see [7]):

Theorem 2 (Goldman-Joichi-White)**.** *Let $B \subseteq [n] \times [n]$ be a proper board. Define the graph*

$$\Gamma(B) = (\mathcal{V}, \mathcal{E}) = ([n], \{(i,j) \notin B : i < j\}) \tag{2}$$

If $C_{\Gamma(B)}$ is the chromatic polynomial of $\Gamma(B)$, then $C_{\Gamma(B)} = \chi_B$.

The next proposition states that for permutation diagrams, we can take the inversion graph instead of the graph Γ of Theorem 2. Moreover, we also show that in such case, the diagram need not be proper. The proposition was stated in [1], for which we should credit Axel Hultman who wrote the appendix where the theorem appears.

Proposition 4 (Hultman)**.** *Let $\sigma \in \mathfrak{S}_n$ be covexillary and G_σ be the inversion graph of σ,* i.e.

$$G_\sigma = (\mathcal{V}, \mathcal{E}) = ([n], \{(i,j) : i < j, \sigma(j) < \sigma(i)\}),$$

Then,

$$C_{G_\sigma}(x) = \chi_{\sigma(x)},$$

where C_{G_σ} is the chromatic polynomial of G_σ.

Sketch of proof.

Step 1: show that the relation stands for σ whenever it stands for $\sigma \ominus \rho_n$.

Step 2: D_τ is a proper board, and thus $\chi_\tau(x) = C_{\Gamma(\tau)}(x)$ by Theorem 2.

Step 3: let $B = \{(i,j) : (i,\sigma(j)) \in D_\sigma\}$, then $B \sim D_\sigma$.

Step 4: $\Gamma(B) = G_\tau(x)$.

$\square$

We can now show that permutations whose rook polynomial has only integer roots are rook-equivalent to a dominant permutation. This is the main result of the section:

Proposition 5. *Let $\sigma \in \mathfrak{S}_n$. Then there exists a dominant permutation τ such that $\tau \sim \sigma$ if and only if χ_σ factors completely in $\mathbb{N}$.*

Proof. The "only if" direction is an immediate consequence of Theorem 1.

For the "if" direction, take a permutation σ for which the rook polynomial χ_σ has only integer roots. Since the rook polynomial is also the chromatic polynomial of the graph $\Gamma(D_\sigma)$, we know that $j \in \mathbb{N}$ is a root of χ_σ if and only if $0 \leqslant j \leqslant c$, where c is the *chromatic number* of Γ. Therefore, the roots of χ_σ form a multiset M of n elements on $[0,c]$, where $\nu(k) > 0$ for all $k \in [0,c]$.

Write M in decreasing order into a tuple $(t_1, \ldots, t_n)$. We then define the tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ where $\lambda_i = n - i - t_i$. We wish to show that λ is the shape of a Ferrers board in $[n] \times [n]$. We can check that λ is a decreasing tuple since we either have $t_{i+1} = t_i$ or $t_{i+1} = t_i - 1$, and in both cases $\lambda_{i+1} \leqslant \lambda_i$. Additionnaly, we have $\lambda_i \leqslant n - i$ since $t_i \geqslant 0$. Because χ_σ is a chromatic polynomial, $\chi_\sigma(0) = 0$ and therefore $t_n = 0$. Because of that $\lambda_n = n - n = 0$. Because λ is decreasing, we have $0 \leqslant \lambda \leqslant n - i$ and λ is the shape of a Ferrers board.

Using Proposition 1, we are able to find $\tau \in \mathfrak{S}_n$ such that D_τ is Ferrers board with shape λ. Using Theorem 1, the roots of χ_τ are exactly the integers $-(\lambda_i - n + i)$. Furthermore, for all i,

$$\nu(-(\lambda_i - n + i)) = \nu(-(n - i - t_i - n + i)) = \nu(t_i),$$

which means that the roots of χ_σ are exactly the roots of χ_τ, and therefore $\chi_\sigma = \chi_\tau$ as desired. $\square$

Foata and Schützenberger have shown in [3] that any Ferrers board is rook-equivalent to a *unique* decreasing Ferrers board. The next proposition was stated as a corollary in [3]:

Proposition 6 (Foata-Schützenberger)**.** *Two decreasing Ferrers boards of respective shape λ and μ are rook-equivalent if and only if $\lambda = \mu$.*

Moreover, because the rook polynomial of a covexillary has only integer roots, we know from Proposition 5 that every covexillary permutation is rook-equivalent to a unique strictly dominant permutation. One can now ask how to find, for any covexillary permutation, the strictly dominant permutation rook-equivalent. Using Proposition 1, we only need to show how to find the shape of the corresponding Ferrers board. The construction we give will make use of this lemma:

Lemma 2. *Let $\sigma \in \mathfrak{S}_n$ be a covexillary permutation and let $c_\sigma = (c_1, \ldots, c_n)$. Then*

$$s_i - n < c_i \leqslant s_i \qquad (1 \leqslant i \leqslant n).$$

Proof. For the first part, observe that $s_i < n$, and thus $s_i - n < 0 \leqslant c_i$ by definition.

For the second part, we suppose $c_i > s_i$. Let $A = \{j : l_j > l_i \text{ or } l_j = l_i\}$. Since the order defined of the rows of D_σ is linear, $\#A = n - s_i$. Letting $k = n - s_i$, we realize that, since

$$j \in A \Rightarrow l_i \subseteq l_j,$$

there are at least k rows l_j for which $c_j > s_i = n - k$. Then by the pigeonhole principle, there must exists an index m such that $m \geqslant k$ and $c_m > n - k$. But by a preceeding lemma,

$$c_m \leqslant n - m \leqslant n - k, \tag{3}$$

yielding a contradiction. Therefore, $s_i - n < c_i \leqslant s_i$ as desired. □

We now give an explicit way of constructing an increasing Ferrers board equivalent to the diagram of a covexillary permutation:

Proposition 7. *Let $\sigma \in \mathfrak{S}_n$ be a covexillary permutation with $c_\sigma = (c_1, \ldots, c_n)$. Let $A = \{c_i - s_i\}$ and define on A the multiset $M = \{(a, \nu(a)) : a \in A\}$, where*

$$\nu(a) = \#\{i \in [n] : c_i - s_i = a\}$$

Then, define the tuples

$$T = \prod_{j=0}^{n-1} (-j)^{\nu(-j) - \mathbf{1}_M(-j)}, \qquad U = \prod_{k=1}^{n} (k-n)^{\mathbf{1}_M(k-n)} \tag{4}$$

Let $TU = (v_1, \ldots, v_n)$, the following tuple is the shape of a strictly Ferrers board B:

$$\lambda = (v_1 + n - 1, \ldots, v_n + n - n),$$

with $B \sim D_\sigma$.

Proof. Because of Lemma 2, we know that $-n < c_i - s_i \leqslant 0$. Therefore, using Proposition 3:

$$\chi_\sigma(x) = \prod_{s=1}^{n} (x + s - n)^{\nu(s-n)}$$

We wish to show that TU is an ordering of M in the following sense: if $T = (t_1, \ldots, t_l)$ and $U = (u_1, \ldots, u_m)$, then

$$\forall a \in A, \#\{j : t_j = a\} + \#\{j : u_j = a\} = \nu(a)$$

Take $a \in A$. Then by Lemma 2, $-n < a \leqslant 0$ and $0 \leqslant -a \leqslant n - 1$. Therefore we can write

$$\#\{j : t_j = a\} = \nu(a) - \mathbf{1}_M(a) = \nu(a) - 1 \tag{5}$$

For the same reason, $1 \leqslant a+n \leqslant n$ and $\#\{j : u_j = a\} = \mathbf{1}_M(a) = 1$. Therefore, we have

$$\#\{j : t_j = a\} + \#\{j : u_j = a\} = \nu(a) - 1 + 1 = \nu(a)$$

for all $a \in A$. Moreover, for any $s \notin A$, both $\nu(s)$ and $\mathbf{1}_M(s)$ are zero, and s is not in T or U by definition.

That being said, we want to show that λ is a decreasing partition. Remark that, by definition, we can write λ in two parts:

$$\lambda = (t_1 + n - 1, \ldots, t_l + n - l)(u_1 + n - (l+1), \ldots, u_m + n - (l+m))$$

By definition of T, we have that $t_i \geqslant t_{i+1}$. Therefore, $t_i + n - i > t_{i+1} + n - (i+1)$ and λ is strictly decreasing on its first part. Additionally we know by Proposition 3 that $\nu(-k) > 0$ whenever $\chi_\sigma(k) = 0$, and since by Proposition 4 χ_σ is a chromatic polynomial, it implies that $\nu(-j) > 0$ for all $j \in [k]$. Thus we have $U = (-m+1, -m+2, \ldots, 0)$. Because $m + l = n$:

$$u_i + n - (l+i) = -m + i + n - l - i = 0,$$

for all i. As U is increasing and $a \in U$ for all $a \in A$, we find that $-m+1 = \min(A)$. So we have

$$t_l + n - l \geqslant -m + 1 + n - l \geqslant -m + n - l - 1 = 0,$$

and because the first part of λ is strictly decreasing, the positivity of λ follows. Besides, since $v_i \leqslant 0$, we have $\lambda_i = v_i + n - i \leqslant n - i$. So we know for a fact that λ is the shape of a decreasing Ferrers board $B \subseteq [n] \times [n]$. Furthermore, because the tuple $(\lambda_1 - n + 1, \ldots, \lambda_n - n + n)$ is the tuple TU, which is a reordering of M, it follows from Theorem 1 that $B \sim D_\sigma$, completing the proof. □

Example 3. *Take the covexillary permutation* $\sigma = (7\,1\,3\,6\,2\,4\,5)$. *The code of* σ *is* $c_\sigma = (0,5,3,0,2,1,0)$ *and the values* s_i *for* $1 \leqslant i \leqslant 7$ *are* $(2,6,5,1,4,3,0)$. *The multiset on* $\{c_i - s_i\}$ *is* $\{(0,1), (-1,2), (-2,4)\}$, *and we have the two following tuples:*

$$T = (-1,-2,-2,-2), \qquad U = (-2,-1,0)$$

The shape of the corresponding decreasing Ferrers board is the tuple

$$\lambda = (5,3,2,1,0,0,0)$$

Now using the recursive formula of Proposition 1, we compute the successive values of the dominant permutation τ*:*

$$\begin{aligned}
\tau(1) &= \max\{j \in [7-5]\} = 2\\
\tau(2) &= \max\{j \in [7-3], j \neq 2\} = 4\\
\tau(3) &= \max\{j \in [7-2], j \neq 2,4\} = 5\\
\tau(4) &= \max\{j \in [7-1], j \neq 2,4,5\} = 6\\
\tau(5) &= \max\{j \in [7], j \neq 2,4,5,6\} = 7\\
\tau(6) &= \max\{j \in [7], j \neq 2,4,5,6,7\} = 3\\
\tau(7) &= \max\{j \in [7], j \neq 2,3,4,5,6,7\} = 1
\end{aligned}$$

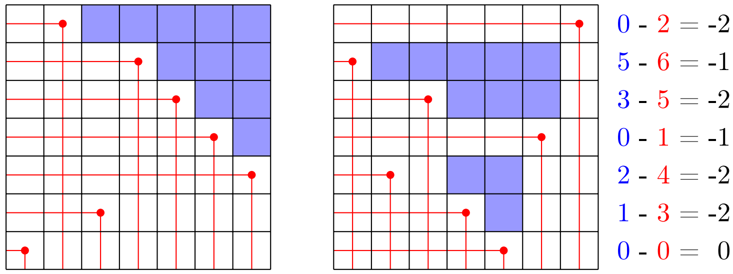

Figure 3: To the right is the diagram of $\sigma = (\,7\ 1\ 3\ 6\ 2\ 4\ 5\,)$ along with the values $c_i - s_i$. To the left is the stricly dominant permutation equivalent to σ.

The permutation $\tau = (\,2\ 4\ 5\ 6\ 6\ 3\ 1\,)$ is strictly dominant and is rook-equivalent to σ. Both permutations can be seen in Figure 3

4 Rook-equivalence in $\mathfrak{S}_n$

Let $\mathcal{R}_n$ be the quotient set of $\mathfrak{S}_n$ by rook-equivalence, that is the set of all rook-equivalence classes in $\mathfrak{S}_n$. We will denote by $[\sigma]$ the class of σ in $\mathcal{R}_n$. The purpose of this section is to give some results and conjectures about this set.

Using results of the preceeding section, we are able to count the number of dominant classes, that is classes containing dominant permutations. More precisely:

Proposition 8. *Let $\mathcal{F}_n = \{[\sigma] \in \mathcal{R}_n : \sigma \text{ is dominant}\}$. Then $\#\mathcal{F}_n = 2^{n-1}$.*

Proof. We know from Proposition 6 that we can index dominant classes using the strictly dominant permutation they contain: if σ and τ are both strictly dominant, then $[\sigma] \neq [\tau]$ whenever $\sigma \neq \tau$. Moreover, because of Proposition7, we know that every dominant class contain such an element. Therefore, it suffices to show that there exist 2^{n-1} strictly dominant permutations in $\mathfrak{S}_n$.

By Proposition 1 and Lemma 1, we only need to prove that the number of decreasing Ferrers boards $B \subseteq T_n$ is 2^{n-1}. We know that a decreasing Ferrers boards always have k non-empty rows with $0 \leqslant k \leqslant n-1$, and at most $n-1$ columns. Therefore, there are $\binom{n-1}{k}$ Ferrers boards $B \subseteq T_n$ with k non-empty lines, as we have to choose the k columns where B will be decreasing. It implies that

$$\#\mathcal{F}_n = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$$

□

So far, we don't have any expression for $\#\mathcal{R}_n$, but we observe that $\#\mathcal{F}_n$ seems to grow much slower than $\#\mathcal{R}_n$. For instance, we know that for $n = 5$, we have 16 dominant classes, over a total of 22 classes, wheras for $n = 9$ there are only 256 dominant classes over a total of 6109. This might be a consequence

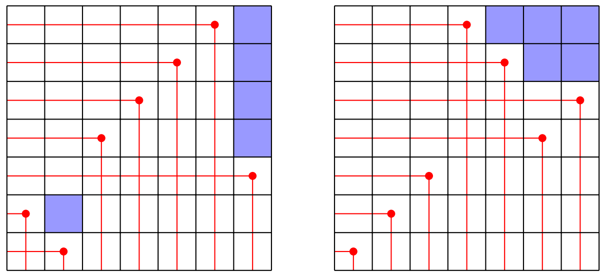

Figure 4: A non-covexillary member of the first mixed class and its strictly dominant equivalent.

of the Stanley-Wilf theorem (see [10]), which in short says that there exists a value c such that the number of covexillary permutations in $\mathfrak{S}_n$ is less than or equal to c^n. However, this is not an immediate corollary, as we do not know how many permutations can be in the same equivalence class.

Based on this evidence, we give the following conjecture:

Conjecture 1. *Let $\mathcal{R}_n$ be the set $\mathfrak{S}_n$ modulo rook-equivalence, and $\mathcal{F}_n$ the set of all classes in $\mathcal{R}_n$ containing a dominant permutation. Then*

$$\lim_{n\to\infty} \frac{\#\mathcal{F}_n}{\#\mathcal{R}_n} = 0.$$

Proposition 3 implies that the rook polynomial of a covexillary permutation has only integer roots. We found that the converse is not true: there are non-covexillary permutations whose rook polynomial factors in $\mathbb{N}$. For instance, the non-covexillary permutation $\sigma = (\,6\,5\,4\,3\,7\,1\,2\,)$ has the following rook polynomial:

$$\sigma(x) = x(x-1)(x-2)(x-3)^3(x-4).$$

By Proposition 5, it follows that there are non-covexillary permutations rook-equivalent to covexillary permutations. An equivalence class containing both covexillary and non-covexillary members is a *mixed class*. We verified that there are no mixed classes in $\mathcal{R}_n$ up to $n = 6$.

Example 4. *We illustrate in Figure 4 one of the four non-covexillary members of the only mixed class of $\mathcal{R}_7$ along with its strictly dominant equivalent. If ρ_n is the nth order reversing permutation, then observe that the non-covexillary member showed in Figure 4 can be written as a skew sum of τ and σ where τ and σ are themselves direct sums of the form $\rho_k \oplus \rho_1$.*

We now give a method to construct an infinite collection of mixed classes. First, we need the following lemma:

Lemma 3. *Let $\tau \in \mathfrak{S}_{m+1} = \rho_m \oplus \rho_1$ (that is, D_τ is a $m \times 1$ vertical rectangle). Then for any permutation $\sigma \in \mathfrak{S}_n$,*

$$\chi_{\sigma\ominus\tau}(x) = (x)_{m+1}\chi_\sigma(x-m-1) + m(x)_m\chi_\sigma(x-m).$$

Proof. We first observe that

$$(x)_k(x-k)_l = \prod_{i=0}^{k-1}(x-i)\prod_{j=0}^{l-1}(x-k-j) = \prod_{i=0}^{k+l-1}(x-i) = (x)_{k+l}. \qquad (6)$$

Also recall the convolution formula of Proposition 2:

$$r_k(\tau \ominus \sigma) = \sum_{i+j=k} r_i(\tau)r_j(\sigma).$$

We also know that $r_0(\tau) = 1, r_1(\tau) = m$ and $r_k(\tau) = 0$ if $k > 1$. Therefore, we have

$$r_k(\tau \ominus \sigma) = r_k(\sigma) + m \cdot r_{k-1}(\sigma),$$

whenever $k \neq 0$.

Suppose $\sigma \in \mathfrak{S}_n$. Using the last relation in the rook polynomial, we find

$$\begin{aligned}
\chi_{\tau\ominus\sigma}(x) = &\sum_{k\geqslant 0} r_k(\tau \ominus \sigma) \cdot (x)_{n+m+1-k} \\
=&(x)_{n+m+1} + \sum_{k\geqslant 1} \big(r_k(\sigma) + m \cdot r_{k-1}(\sigma)\big)(x)_{n+m+1-k} \\
=&(x)_{n+m+1} + \sum_{k\geqslant 1} r_k(\sigma) \cdot (x)_{n+m+1-k} \\
&+ m(x)_{n+m} + m \sum_{k\geqslant 1} r_k(\sigma) \cdot (x)_{n+m-k}.
\end{aligned}$$

Using Equation 6, we are able to rearrange the last expression to obtain:

$$\begin{aligned}
\chi_{\tau\ominus\sigma}(x) =&(x)_{m+1}\left((x-m-1)_n + \sum_{k\geqslant 1} r_k(\sigma)(x-m-1)_{n-k}\right) \\
&+ m(x)_m\left((x-m)_n + \sum_{k\geqslant 1} r_k(\sigma) \cdot (x-m)_{n-k}\right) \\
=&(x)_{m+1}\chi_\sigma(x-m-1) + m(x)_m\chi_\sigma(x-m)
\end{aligned}$$

as desired. □

We now give our construction:

Proposition 9. *Let $\tau \in \mathfrak{S}_{m+1}$ be the direct sum $\rho_m \oplus \rho_1$, and $\sigma \in \mathfrak{S}_{n+1}$ the direct sum $\rho_n \oplus \rho_1$. Let $\omega = \sigma \ominus \tau$. If $n = 2m+2$, then $[\omega] \in \mathcal{R}_{n+m+2}$ is a mixed class, ω itself being one of its non-covexillary elements.*

Proof. We know that $\chi_\sigma(x) = (x)_{n+1} + n(x)_n$. Using this with Lemma 3, we have

$$\chi_\omega(x) = (x)_{m+1}\big((x-m-1)_{n+1}+n(x-m-1)_n\big)+m(x)_m\big((x-m)_{n+1}+n(x-m)_n\big).$$

Using again Equation (6) yields

$$\chi_\omega(x) = (x)_{n+m+2} + (n+m)(x)_{n+m+1} + mn(x)_{n+m}.$$

Factoring $(x)_{n+m}$, we find

$$\chi_\omega(x) = (x)_{m+n}\big((x-m-n)(x-m-n-1) + (n+m)(x-m-n) + mn\big).$$

Assuming $n = 2m+2$, we can simplify to

$$\chi_\omega(x) = (x-m-2)(x-2m-1)(x)_{3m+2}$$

and therefore χ_ω factors in $\mathbb{N}$. Furthermore, we can prove that ω is non-covexillary. Using the definitions of skew and direct sum, we compute:

$$\omega(1) = (\rho_{2m+2} \oplus \rho_1)(1) + m + 1 = \rho_{2m+2}(1) + m + 1 = 3m+3$$
$$\omega(2m+3) = (\rho_{2m+2} \oplus \rho_1)(2m+3) + m + 1 = \rho_1(1) + 3m + 3 = 3m+4$$
$$\omega(2m+4) = (\rho_m \oplus \rho_1)(1) = \rho_m(1) = m$$
$$\omega(3m+4) = (\rho_m \oplus \rho_1)(m+1) = \rho_1(1) + m = m+1,$$

Thus, the sequence $(\omega(1), \omega(2m+3), \omega(2m+4), \omega(3m+4))$ is order-isomorphic to $(\,3\ 4\ 1\ 2\,)$ and ω is non-covexillary as desired. □

The proof of Proposition 9 is constructive, so we can give an example of such a mixed class:

Example 5. *Fix $m = 3$ and take*

$$\omega = (\rho_8 \oplus \rho_1) \ominus (\rho_3 \oplus \rho_1) = (\,12\ 11\ 10\ 9\ 8\ 7\ 6\ 5\ 13\ 3\ 2\ 1\ 4\,).$$

Then, using Equation (4), we can factor χ_ω immediately:

$$\chi_\omega = (x-5)(x-7)(x)_{11}.$$

According to Proposition 7, the corresponding strictly dominant permutation is the one whose diagram has shape $\lambda = (7, 4, 0 \ldots, 0)$ in $\mathfrak{S}_{13}$. We illustrate both permutations at Figure 5.

One can easily see that for any m, the strictly dominant permutation $\sigma \in [\omega]$ will satisfy

$$c_\sigma = (2m+1, m+1, 0, \ldots, 0).$$

We now give a conjecture concerning mixed classes:

Conjecture 2. *Let $[\omega] \in \mathcal{R}_n$ be a mixed class. Then for any permutations $\sigma \in [\omega]$, σ contains the pattern $(\,3\ 4\ 1\ 2\,)$ $4k$ times, $k \in \mathbb{N}$.*

This conjecture has been verified in $\mathcal{R}_n$ up to $n = 9$. Furthermore, we can prove that the non-covexillary permutation constructed using Proposition 9 contains the pattern $(\,3\ 4\ 1\ 2\,)$ $4k$ times. Indeed, it suffices to show that the number of unordered tiles $m(2m+2) \equiv 0 \mod 4$, which is fairly straightforward.

We have already checked that the converse of the conjecture is false. That is, containing the pattern $(\,3\ 4\ 1\ 2\,)$ $4k$ times, $k \in \mathbb{N}$, might be a necessary but insufficient condition for a non-covexillary permutation to be in a mixed class. We wonder what a necessary *and* sufficient condition would be.

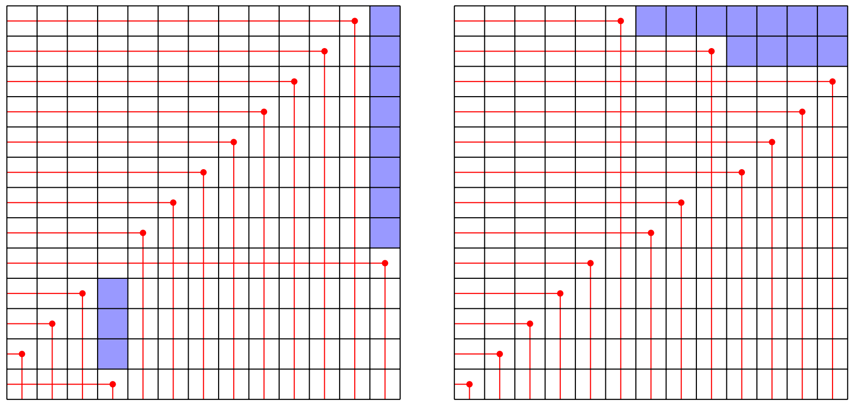

Figure 5: A non-covexillary member of a mixed class constructed using Proposition 9, along with its strictly dominant equivalent.

5 A labelling of Ferrers board

The purpose of this section is to give a labelling of Ferrers board (or, equivalently, of dominant permutations) that regroups many interesting properties. For instance, this labelling gives a factorization of the rook polynomial, while caracterising the equivalence class of the diagram. It also tells us the greatest number of rooks we can place on this diagram, while identifying a "canonical" k rooks placement. We think this labelling could be used to build an explicit bijection between rooks placements on equivalent Ferrers boards.

Definition 4. *Let $\sigma \in \mathfrak{S}_n$ be a dominant permutation. Let $D'_\sigma = D_\sigma \cup \{(i, n+1) : i \in [n]\}$ be the right extension of D_σ. That is, D'_σ is the board obtained by adding a n column at the right of D_σ. We define the map $\Phi_\sigma : D'_\sigma \to \mathbb{N}$ by*

$$\Phi_\sigma : (i, j) \mapsto n + i - j.$$

Φ_σ is called the labelling of D_σ, whereas $\Phi_\sigma(t)$ is called the label of t.

Example 6. *One can compute the labelling of a Ferrers board by filling all the tiles in the kth principal diagonal (starting in the upper right corner) with k's. We illustrate the labelled diagram of the dominant permutation $\sigma = \left(\begin{smallmatrix} 3 & 5 & 4 & 2 & 7 & 6 & 8 & 1 \end{smallmatrix} \right)$ in Figure 6.*

This particular labelling has many interesting properties. For instance, we can rewrite Theorem 1 using the labelling function:

Proposition 10. *If $\sigma \in \mathfrak{S}_n$ is dominant, let*

$$M_\sigma = \{(i, j) : (i, k) \in D_\sigma \Rightarrow \Phi_\sigma(i, k) \leqslant \Phi_\sigma(i, j)\}.$$

Then:

$$\chi_\sigma(x) = \prod_{t \in M_\sigma} (x + \Phi_\sigma(t) - n + 1).$$

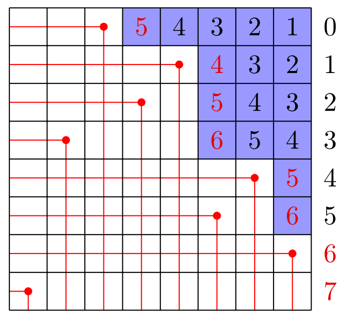

Figure 6: The labelled diagram of $\sigma = (3\,5\,4\,2\,7\,6\,8\,1)$. The red labelled cells are those for which the labelling function is maximal on the corresponding row.

Proof. Because σ is dominant, every tile $t \in M$ is always the leftmost tile of D'_σ. If t is the leftmost tile of the ith row, it follows that $t = (i, n - c_i + 1)$, and therefore

$$\Phi_\sigma(t) - n + 1 = n + i - n + c_i - 1 - n + 1 = c_i - n + i.$$

Thus, by Theorem 1, the proposition stands. □

Example 7. *Coming back to Example 6, we can use the labelled diagram to compute the rook polynomial. In Figure 6, the labels of the cells $t \in M_\sigma$ are shown in red. Using the last proposition, we compute row by row:*

$$\chi_\sigma(x) = (x-2)(x-3)(x-2)(x-1)(x-2)(x-1)(x-1)x = x(x-1)^3(x-2)^3(x-3).$$

In [2], Ding remarked that, for any Ferrers board B, there exists a "canonical" k-rook placement, in the sense that this rook placement is always in $R_k(B)$ whenever $R_k(B)$ is non-empty. This canonical rook placement can be characterized by our labelling function:

Proposition 11. *For every dominant permutation σ, $R_k(\sigma) \neq \varnothing$ implies that $R \in R_k(\sigma)$ where*

$$R = \{t \in D_\sigma : \Phi_\sigma(t) = k\}.$$

Thus we can place k non-attacking rooks on D_σ if and only if k tiles of D_σ are labelled k.

Proof. We first remark that the set $\{t \in D_\sigma : \Phi_\sigma(t) = k\}$ is a k-rook placement on D_σ for all k (with the convention that $R_0(\sigma) = \{\varnothing\}$). Indeed, as we fix alternatively i or j, there exists only one solution to the equation $\Phi_\sigma(i, j) = k$. Therefore we have at most one tile labelled k for a given row or column.

The "if" direction is then immediate, for if k tiles in D_σ are labelled k, the rook placement $\{t \in D_\sigma : \Phi_\sigma(t) = k\}$ is a k-rook placement.

For the "only if" direction, we need only to show that $R_k(\sigma) \neq \varnothing$ implies that at least k tiles in D_σ are labelled k. Begin by taking a k-rook placement

R over D_σ. For $1 \leqslant i \leqslant k$, let a_i be the number of rooks in R that are above or on the ith row, and b_i be the number of rooks in R below the ith row. It follows that $a_i + b_i = k$. Suppose that the number c_i of tiles in row i satisfies $c_i = b_i$. Then we must have $a_i < i$ for there can be no rooks in the ith row. Therefore $b_i > k - i$ and we have $(i, n - k + i) \in D_\sigma$. On the other hand, suppose that $c_i > b_i$. Then, because $a_i \leqslant i$, we have $c_i > b_i \geqslant k - i$ and, again, $(i, n - k + i) \in D_\sigma$. Labelling this particular tile yields:

$$\Phi_\sigma(i, n - k + i) = n + i - n + k - i = k,$$

as desired. □

Example 8. *Taking again* $\sigma = (3\,5\,4\,2\,7\,6\,8\,1)$, *we see directly from Figure 6 that it is impossible to place more than 4 rooks over* D_σ, *because there are less than 5 tiles labelled 5 in* D_σ.

Also remark that we can never have more than k tiles labelled k in D_σ, for the kth principal diagonal of the grid (starting from the upper right corner) has k elements.

It follows directly from the two last propositions that rook-equivalence for dominant permutations may be reduced to isomorphism of the labelling functions. To show this, we first need a lemma:

Lemma 4. *Let* σ *and* τ *be two dominant permutations such that* $\sigma \sim \tau$. *Then the maximum of* Φ_σ *restricted to* D_σ *is equal to the maximum of* Φ_τ *restricted to* D_τ.

Proof. Call m the maximum of Φ_σ restricted to D_σ and l the maximum of Φ_τ restricted to D_τ. With no loss of generality, we can assume $m > l$. Because $\sigma \sim \tau$, the $(m+1)$th row of D_τ must be empty, or else $(x + m - n + 1)$ would not be a factor of p_τ whereas it will be one of p_σ, and that would contradict the fact that $\sigma \sim \tau$.

But it is also clear that $(x + m - n + 1)^2$ is not a factor of p_τ because $m + 1$ is the only solution to $\Phi_\tau(x, n + 1) = m$. It follows that the $(m+1)$th row of p_σ must not be empty, and therefore we must have $(m + 1, n) \in D_\tau$. Labelling this tile yields

$$\Phi_\sigma(m + 1, n) = n - n + m + 1 = m + 1,$$

which contradict the fact that m is the maximum of Φ_σ over D_σ. Therefore $m = l$ as desired. □

We can now prove the following proposition:

Proposition 12. *Two dominant permutations* $\sigma, \tau \in \mathfrak{S}_n$ *are equivalent if and only if* $\Phi_\sigma \cong \Phi_\tau$, *in the sense that there exists a bijection* $\psi : D_\sigma \to D_\tau$ *such that*

$$\Phi_\sigma = \Phi_\tau \circ \psi \tag{7}$$

Proof. We proceed by induction over the number of tiles s of the boards. Our base case, $s = 2$, is easy because the only dominant permutations in $\mathfrak{S}_n$ with 2 tiles are those with shape $(2, 0, \ldots, 0)$ and $(1, 1, 0, \ldots, 0)$. They are rook-equivalent, and we indeed have a bijection ψ defined by

$$\psi(1, n) = (1, n), \quad \psi(1, n-1) = (2, n),$$

that satisfies Equation (7) for those two permutations. Therefore the proposition stands for every dominant permutation whose diagram has two tiles.

Take σ and τ in $\mathfrak{S}_n$ with s tiles (assuming $s \geqslant 2$). By induction we assume any two dominant permutations σ', τ' with less than s tiles are rook equivalent if and only if we can find a bijection from $D_{\sigma'}$ to $D_{\tau'}$ satisfying Equation (7).

For the "if" direction, suppose there exists a bijective map $\psi : D_\sigma \to D_\tau$ satisfying Equation (7). Then the maximum of both labelling functions Φ_σ, Φ_τ over D_σ and D_τ is the same. Let m be that value and take any tile $t = (i, j)$ in D_σ labelled m. By definition of ψ, $\psi(t)$ is also labelled m. Because m is maximal, both t and $\psi(t)$ must be the leftmost tile of some row in D_σ and D_τ, and therefore we can use Proposition 10 to say that

$$\chi_\sigma(x) = (x + m - n + 1)P(x), \quad \chi_\tau(x) = (x + m - n + 1)Q(x), \tag{8}$$

where P and Q are two polynomials.

Remark that because m is maximal, there can be no cell in D_σ South of t, for $\Phi_\sigma(t) = m$ would then not be maximal. The same remark also applies to $\psi(t)$ in D_τ. It follows that both $D_\sigma - \{t\}$ and $D_\tau - \{\psi(t)\}$ are Ferrers board. Because they are respective subsets of D_σ and D_τ, Proposition 1 tells us we can find two dominant permutations $\tau', \sigma' \in \mathfrak{S}_n$ satisfying

$$D_{\sigma'} = D_\sigma - \{t\}, \quad D_{\tau'} = D_\tau - \{\psi(t)\} \tag{9}$$

Using Proposition 10 again, the rook polynomial of σ' and τ' can easily be expressed in terms of the polynomials P and Q of Equation (8), for we know that $D_{\sigma'}$ and $D_{\tau'}$ differ respectively from D_σ and D_τ only by the row on which t, respectively $\psi(t)$, lies. Moreover, for this particular row, the leftmost cell is now labelled $m - 1$, and thus

$$\chi_{\sigma'}(x) = (x + m - n)P(x), \quad \chi_{\tau'}(x) = (x + m - n)Q(x).$$

By construction, the restriction of ψ to $D_\sigma - \{t\}$ still satisfies Equation (7), and thus, by induction, $\sigma' \sim \tau'$. It follows that $P = Q$, and therefore, because of Equation (8), we also have $\sigma \sim \tau$.

For the "only if" direction, we now suppose that σ and τ are rook-equivalent. By Lemma 4, call m the maximum of both Φ_σ restricted to D_σ and Φ_τ restricted to D_τ. Take two tiles t, u respectively in D_σ and D_τ such that $\Phi_\sigma(t) = \Phi_\tau(u) = m$. Using Proposition 1, we can find two dominant permutations $\sigma'\tau' \in \mathfrak{S}_n$ such that

$$D_{\sigma'} = D_\sigma - \{t\}, \quad D_{\tau'} = D_\tau - \{u\}$$

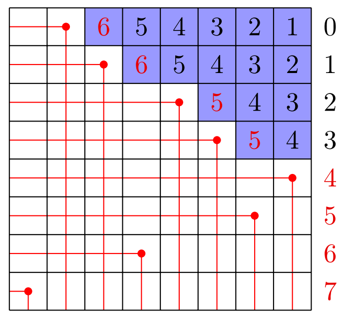

Figure 7: The labelled diagram of $\tau = (2\,3\,5\,6\,8\,7\,4\,1)$.

As we already said, the set $M_{\sigma'}$ equals M_σ except for the row where we remove the tile t. As the same remark applies for τ' and τ, we can give both the polynomials of τ' and σ' using Proposition 10:

$$\chi_{\sigma'}(x) = \left(\frac{x+m-n}{x+m-n+1}\right)\chi_\sigma(x), \quad \chi_{\tau'}(x) = \left(\frac{x+m-n}{x+m-n+1}\right)\chi_\tau(x).$$

As we assumed that $\sigma \sim \tau$, we have $\chi_\sigma = \chi_\tau$ and therefore $\sigma' \sim \tau'$. Thus, by induction, we can find a bijective map $\psi' : D_{\sigma'} \to D_{\tau'}$ satisfying Equation (7). Then it suffices to extend such a map to all D_σ by defining

$$\psi(i,j) = \begin{cases} u & \text{if } (i,j) = t \\ \psi'(i,j) & \text{otherwise,} \end{cases}$$

The map ψ is indeed bijective since ψ' is and $u \notin D_{\tau'}$. Moreover, because $\Phi_\sigma(t) = \Phi_\tau(u)$, ψ also satisfies Equation (7). That is, we found the desired map, completing the proof. □

Example 9. *One may compare the labelled diagram of $\tau = (2\,3\,5\,6\,8\,7\,4\,1)$, in Figure 9, with the labelled diagram of Figure 6. Since they have the same number of cells labelled k for all k, we can find a bijective map from one diagram to the other that preserves the labelling. Thus, by the preceeding proposition, τ and the permutation of Example 6 are rook-equivalent.*

Corollary 2. *Let $\sigma, \tau \in \mathfrak{S}_n$ such that $\sigma \sim \tau$. Let m be the maximum of both Φ_σ and Φ_τ, and let $t \in D_\sigma$ and $u \in D_\tau$ such that $\Phi_\sigma(t) = \Phi_\tau(u) = m$. Then for all k, the number of k rooks placements with a rook on t is equal to the number of k rooks placements with a rook on u.*

Proof. Remark that

$$r_k(\sigma) = \#\{R \in R_k(\sigma) : t \in R\} + \#\{R \in R_k(\sigma) : t \notin R\}$$

As the same remark apply for τ and because $r_k(\sigma) = r_k(\tau)$, it suffices to show that

$$\#\{R \in R_k(\sigma) : t \notin R\} = \#\{R \in R_k(\tau) : u \notin R\} \tag{10}$$

We can count these by counting the number of $k-1$ rooks placements on the boards obtained by deleting respectively the cells t and u. Let σ' and τ' be the dominant permutations realizing those boards. By Proposition 12, we know there exists a bijection between D_σ and D_τ that preserves the labelling, and we can restrict it to $D_{\sigma'}$ and $D_{\tau'}$. Again, by Proposition 12, it follows that $\sigma' \sim \tau'$, and therefore $r_{k-1}(\sigma') = r_{k-1}(\tau')$ as desired. □

6 On q-analogues of rook placements

In [6], Haglund established a connection between Rook Theory and $n \times n$ matrices in $\mathbb{F}_q$ with certain entries restricted to zero.The case where the restricted entries have the shape of a permutation diagram have been studied in [8] and [1]. The connection between Rook Theory and restricted matrices over $\mathbb{F}_q$ was made through a q-analogue of the numbers $r_k(B)$ defined by Garsia and Remmel in [4]. We start by giving a formal definition of Garsia and Remmel's q-analogue for rook numbers:

Definition 5. *Let B be a board and take a rooks placement R over B. Let*

$$N_R = \{(i,j) : (i,k) \notin R \text{ if } k \leqslant j, (l,j) \notin R \text{ if } l \geqslant i\}.$$

The Garsia-Remmel number of R as a rook placement over B is

$$\mathrm{GR}(R,B) = \#(N_R \cap B).$$

The q-rook numbers of B are the polynomials

$$r_k(B,q) = \sum_{R \in R_k(B)} q^{\mathrm{GR}(R,B)}.$$

If $B = D_\sigma$ for a certain permutation σ, we will then write $r_k(\sigma,q)$ in place of $r_k(B,q)$ without ambiguity.

Thus, we define a q-analogue of the rook polynomial:

Definition 6. *Let $B \subseteq [n] \times [n]$, we define the q-rook polynomial of B:*

$$\chi_B(x,q) = \sum_{k \geqslant 0} r_k(\sigma,q)[x]_q[x-1]_q \dots [x-(n-k)+1]_q,$$

where $[x]_q = (1-q^x)/(1-q)$. If $B = D_\sigma$ for $\sigma \in \mathfrak{S}_n$, we note $\chi_\sigma(x,q)$.

This q-analogue of Garsia and Remmel is a good q-analogue in many ways. Firstly, it is clear that $r_k(B,1) = r_k(B)$. Furthermore, Garsia and Remmel proved, also in [4], the following formula:

Theorem 3 (Garsia-Remmel)**.** *For any Ferrers board B of shape $\lambda = (\lambda_1, \dots, \lambda_n)$,*

$$\chi_B(x,q) = \prod_{i=1}^{n} [x + \lambda_i - n + i]_q.$$

We now consider q-rook equivalence classes in $\mathfrak{S}_n$. More precisely:

Definition 7. *We shall say that $\sigma \sim_q \tau$ if $r_k(\sigma, q) = r_k(\tau, q)$ for all k. Equivalently, $\sigma \sim_q \tau$ if and only if $\chi_\sigma(x, q) = \chi_\tau(x, q)$.*

In [8], Brewster, Klein and Morales essentially remarked that for covexillary permutations, q-rook-equivalence is the same as ordinary rook-equivalence. Therefore, it is clear that there are also 2^{n-1} dominant classes in $\mathfrak{S}_n$ under q-rook-equivalence. Interestingly, q-rook-equivalence seems to be an helpful tool to deal with mixed classes. Based on our observations, q-rook-equivalence is strong enough to split mixed classes, and precise enough so it regroups the permutations according to the number of times they contain the pattern $(3\,4\,1\,2)$. The following conjecture is verified in $\mathfrak{S}_n$ up to $n = 9$:

Conjecture 3. *Let $\sigma, \tau \in \mathfrak{S}_n$ with σ covexillary and τ non-covexillary. Then $\sigma \not\sim_q \tau$.*

Whenever $\sigma \not\sim \tau$, the conjecture is true, for $\sigma \not\sim \tau$ implies $\sigma \not\sim_q \tau$ as well. Thus, the difficult part of this conjecture is to explain how it tears apart the mixed classes.

Proposition 13. *Conjecture 3 stands if $\tau = (\rho_{2m+2} \oplus \rho_1) \ominus (\rho_m \oplus \rho_1)$.*

Proof. Assume $\tau = (\rho_{2m+2} \oplus \rho_1) \ominus (\rho_m \oplus \rho_1)$ and take $\sigma \in \mathfrak{S}_{3m+4}$ covexillary such that $\sigma \sim \tau$. Since q-rook-equivalence is the same as ordinary rook-equivalence, we can consider without losing generality that σ is dominant. We will show that $r_1(\sigma, q) \neq r_1(\tau, q)$, for in particular the coefficient of q^{c-1}, where $c = \#D_\sigma = \#D_\tau$, is 1 in $r_1(\sigma, q)$ whereas it is 2 for $r_1(\tau, q)$.

Since σ is dominant, it is the shape of a Ferrers board, and therefore placing a rook on the upper right corner of D_τ is the only 1 rook placement for which the Garsia-Remmel number is $c - 1$. Thus the coefficient of q^{c-1} in $r_1(\sigma)$ is 1 as desired.

By definition, τ is a disjoint union of two rectangles. Thus, there is two 1 rook placement for which the Garsia-Remmel number is $c-1$: we can either place one rook on the upper right corner of the first or of the second rectangle. Therefore the coefficient of q^{c-1} is 2 in $r_1(\tau, q)$, and $r_1(\sigma, q) \neq r_1(\tau, q)$, completing the proof. ∎

We now give an example to show how mixed classes behave under q-rook-equivalence:

Example 10. *Let $\sigma = (4\,5\,6\,7\,9\,8\,3\,2\,1)$. The class $[\sigma] \in \mathcal{R}_9$ is a mixed and σ is its strictly dominant element. We spare the reader the inconvenience of an exhaustive list of the 438 elements of $[\sigma]$. Note that*

$$\chi_\sigma(x) = (x)_9 + 14(x)_8 + 55(x)_7 + 65(x)_6 + 16(x)_5.$$

Of the 438 elements of $[\sigma]$, 416 are covexillaries, 20 contain the pattern $(3\,4\,1\,2)$ 4 times, and only 2 contain it 8 times. To exemplify, we give $\tau =$

$r_0(\sigma, q)$	q^{14}
$r_1(\sigma, q)$	$q^{13} + 2q^{12} + 3q^{11} + 4q^{10} + 4q^9$
$r_2(\sigma, q)$	$q^{11} + 3q^{10} + 7q^9 + 12q^8 + 14q^7 + 12q^6 + 6q^5$
$r_3(\sigma, q)$	$q^8 + 4q^7 + 10q^6 + 16q^5 + 18q^4 + 12q^3 + 4q^2$
$r_4(\sigma, q)$	$q^4 + 4q^3 + 6q^2 + 4q + 1$
$r_0(\tau, q)$	q^{14}
$r_1(\tau, q)$	$q^{13} + 2q^{12} + 5q^{11} + 2q^{10} + 2q^9 + 2q^8$
$r_2(\tau, q)$	$q^{11} + 5q^{10} + 10q^9 + 10q^8 + 11q^7 + 10q^6 + 5q^5 + 2q^4 + q^3$
$r_3(\tau, q)$	$4q^8 + 8q^7 + 11q^6 + 14q^5 + 13q^4 + 8q^3 + 5q^2 + 2q$
$r_4(\tau, q)$	$q^6 + 2q^5 + 3q^4 + 4q^3 + 3q^2 + 2q + 1$
$r_0(\omega, q)$	q^{14}
$r_1(\omega, q)$	$q^{13} + 2q^{12} + 4q^{11} + 3q^{10} + 3q^9 + q^8$
$r_2(\omega, q)$	$q^{11} + 4q^{10} + 8q^9 + 12q^8 + 13q^7 + 9q^6 + 6q^5 + 2q^4$
$r_3(\omega, q)$	$2q^8 + 6q^7 + 12q^6 + 15q^5 + 14q^4 + 10q^3 + 5q^2 + q$
$r_4(\omega, q)$	$q^5 + 3q^4 + 4q^3 + 4q^2 + 3q + 1$

Table 1: The respective non-zero q-rook numbers of $\sigma = (4\,5\,6\,7\,9\,8\,3\,2\,1)$, $\tau = (4\,5\,8\,9\,6\,3\,2\,1\,7)$ and $\omega = (3\,8\,7\,6\,5\,1\,9\,2\,4)$.

$(4\,5\,8\,9\,6\,3\,2\,1\,7)$ containing the pattern $(3\,4\,1\,2)$ 4 times, and $\omega = (3\,8\,7\,6\,5\,1\,9\,2\,4)$ containing it 8 times. We illustrate σ, τ and ω in Figure 8.

All the covexillaries elements have the same q-rook polynomial as σ, all the elements containing $(3\,4\,1\,2)$ 4 times have the same rook polynomial as τ and all the elements containing it 8 times have the same rook polynomial as ω. Instead of giving the polynomials (which are fairly long), we list the corresponding q-rook numbers in Table 1.

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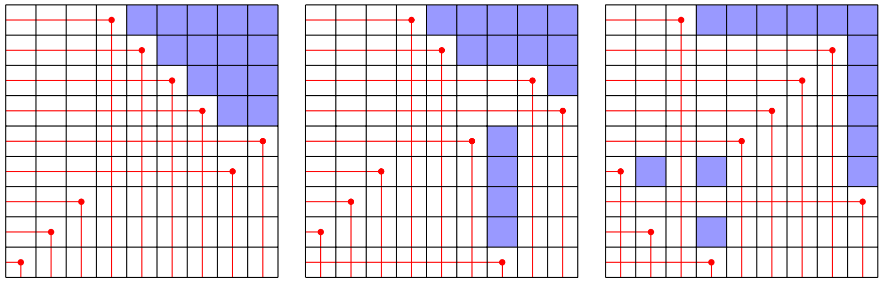

Figure 8: From left to right: the diagram of (4 5 6 7 9 8 3 2 1), (4 5 8 9 6 3 2 1 7) and (3 8 7 6 5 1 9 2 4). The three permutations are rook-equivalent.

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